An overview on properties and efficacy of topological skeletons in Shape Modelling

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Abstract

This paper investigates the main issues related to the definition of abstraction tools for deriving high-level descriptions of complex geometric models. Among the wide range of shape descriptors, topological graph-like representations not only give a powerful and synthetic sketch of the object, but also capture its inner structure, that is how features connect together to give the overall shape. This aspect makes them useful to describe complex 3D objects in various applications like modelling, morphing, matching, and recognition. The paper surveys the main properties of skeletons developed in Shape Modelling for representing objects.

1. Introduction

The process of extracting a compact and expressive description from digital models of three-dimensional objects is useful in many applications such as recognition, classification, modification, approximation, retrieval of 3D shapes and all those tasks that are computationally expensive if performed on huge data sets.

For the generation of high-level descriptions it is useful to consider the shape as a combination of geometry, topology (structure) and semantic, [14]. We can perceive the differences between form and structure by thinking about the object shape from an intuitive point of view. In this sense, the form of an object can be described by adjectives such as long, short, flat, rounded, squared, conic, cylindrical, spherical, etc.; furthermore, when the object is complex, it may have holes, protrusions and concavities. The form concept is understood as a local property of the object, while structure deals with the whole object and concerns its global features and how they are related to each other. From this point of view, an object can be partitioned into protrusions, holes and other characteristics and can be efficiently represented as a collection of morphological features with a set of adjacency relations between them. These facts raise the idea that topology-based descriptors, maybe integrated with geometric information, are suitable for dealing with the definition of basic models for representing, generating and manipulating shapes without forgetting the feasibility and the computational complexity of the problem, [12][18].

Moreover, shape understanding is especially relevant for the perception of complex shapes, where the ability of varying the level of descriptive abstraction is the key for recognizing and classifying them, [29]. This aim can be achieved through a multi-resolution framework that abstracts the object at different resolutions, locating the main features and discarding details, without losing the overall appearance, [14].

The remainder of the paper is organized as follows. First, the effective application of structural graphs to computer graphics is discussed and, in Section 3, two skeleton-like representations, the Medial Axis and the Reeb graph, are presented. Then, a detailed overview of several Reeb graph representations is given. Discussions on the advantages and limits of using skeletons in shape modeling conclude the paper.

2. Effectiveness of topological graphs

As mentioned above, the best advantage of high-level shape descriptors is to capture the relevant features of the shape and the relations between them, and to code and preserve the topological properties of
the object, as the number of connected components and holes. In this way, the information used to describe the shape of the object is strongly reduced with respect to the whole model, [14].

The use of skeletal descriptors is best suited for accomplishing tasks like shape retrieval, recognition and alignment, and the quality of the result depends on the effectiveness of the chosen description. In particular, shape retrieval deals with the matching of an input object with a huge database of models for extracting the most similar one through a large number of comparison operations. The aim of shape recognition is to find out, from a set of object classes, the one the input shape belongs to. Finally, shape alignment intends to determine a sequence of transformations (editing operations) that maximize the similarity between two objects. Despite the information needed to describe the shape, all these kinds of applications are related to the basic problem of estimating the similarity between shapes. Commonly, methodologies to achieve this goal return a numerical value but are unable to explain why two objects are similar or dissimilar [26][28] while new trends aim to deduce the similarity measure directly from the skeleton [19].

A similarity measure should satisfy the properties of non-negativity, identity, uniqueness and triangular inequality. When all these properties hold, the similarity measure is a metric, while if the uniqueness is lost the measure is a semi-metric, [35]. From an intuitive point of view, a negative similarity measure does not make sense, while the identity property claims that a shape is similar (identical) to itself. Instead, the uniqueness property ensures that, when the dissimilarity is zero, the two graphs are isomorphic. This property is very strong for a high-level shape descriptor comparison, and often it is not satisfied. However, this is not a relevant drawback, because the main goal of high-level shape descriptors is to represent only the relevant features of the shape, and the loss of uniqueness depends on negligible details. The triangular inequality is a very useful property in applications based on shape retrieval and, together with the identity, returns the symmetric property, [35]. Another important property that a similarity measure should satisfy is the invariance from rotations, translations and scaling operations, because the comparison and the extraction process of high-level descriptors have to be independent on the coordinate system and robust to small perturbations of the object shape. This means that localized and irrelevant changes with respect to the main features of the shape do not strongly affect the similarity measure and the matching process.

Another application field of topological graphs is object reconstruction and editing where the stored information is exploited for recovering, approximating and modifying the input geometry. For instance, the knowledge of the shape topology given by the graph structure, improves the tiling from contour lines, [6] thus solving the correspondence and the branching problems, [16][23][27]. Topological graphs provide an approximation of the input object useful for recovering a sketch of the original surface whose quality depends on the stored information and on the underlying recovering algorithm (i.e. implicit surfaces based on convolution [9][5] or radial basis functions [34], direct methods [1]). For instance, [1] uses skeletons for defining an approximation of the input surface geometry by using basic primitives, i.e. generalized cylinders, each one related to building elements of the skeleton such as triangles, edges or vertices with the local information stored during the graph construction.

In implicit modelling, skeletons, considered as a collection of elements with associated implicit primitives, provide a compact representation that is useful in defining both motion and deformation. Typical skeletons are hierarchical (in the sense that they reflect the intuitive layered construction of the shape) and are usually represented as a direct, acyclic graph. Often the medial axis, eventually pruned to eliminate artefacts introduced by noise [2], is chosen as the skeletal element, and has been proposed for volumetric restoring purposes [8]. In that case, skeletons may consist of a set of geometric primitives such as points, curves, polygons, etc. Skeleton-like structures are essential for implicit model animation; in fact, during animation, its attributes may change, varying, for instance, radius, blending and other surface details. Moreover, depending on the kind of the manipulation task (animation, metamorphosis, growth, etc.), skeletal elements may rotate, stretch, appear or disappear. However, the skeletal elements of the intermediate shapes obtained during the animation evolution remain simply to define, articulate and display and the skeletal hierarchy (that is the internal relationships between arcs and nodes) generally does not change [5].

We show in the following that an optimal descriptor for all applications is still lacking. We describe several topological graphs together with
their properties, underlining the tasks for which each of them is more suitable.

3. Two examples: the MAT and the Reeb Graph

Among the existing techniques for extracting the structure of a shape, the Medical Axis (MA) is generally considered the most elegant and effective one. An intuitive definition of the skeleton in the continuum was given by Blum [9], who described the skeleton extraction as a fire front which starts at the boundary of the shape and propagates isotropically towards the interior. The medial axis is defined by the locations at which the fire fronts collide.

More formally, the medial axis of a shape $S$ in $\mathbb{R}^n, n \geq 2$, is the locus of centres of all maximal discs\(^1\) of $S$, that is, those discs contained in $S$ which are not contained in any other disc in $S$ (see Figure 1(a)). Equivalently, if $B(S)$ is the boundary of $S$, then the medial axis of $B(S)$ is the set of points in $S$ having at least two nearest neighbours on $B(S)$. In practice, we can associate to every point $x$ of the figure its distance from the set $B(S), d(x, B(S))$, defined as $d(x, B(S)) = \inf \{d(x, y) : y \in B(S)\}$. However, there are some points where the distance is not achieved uniquely: for such points $x$, at least two boundary points ($y$ and $z$) can be found such that $d(x, B(S)) = d(x, y) = d(x, z)$. These “singular” points $x$ define the nodes of the graph which correspond either to areas where the shape branches or end parts of protrusion-like structures (Figure 1(b)). The medial axis, together with the radius function, i.e., the distance from each point on the axis to the nearest point on the boundary, defines the Medical Axis Transform (MAT).

The power of this representation is that the shape boundary and its MAT are equivalent and the one can be computed from the other [9] (the original shape can be recovered from its medial axis using a simple distance transform); therefore, a two-dimensional object is effectively compressed into a one-dimensional graph-like structure.

If the shape is a polygon, the MAT is a tree-like planar graph whose arcs are composed by straightline segments and portions of parabolic curves; several algorithms have been defined to compute the MAT from the Voronoi diagram of the polygon.

\(^1\) From a mathematical point of view a disc is a subset of $\mathbb{R}^n$ homeomorphic to the $n$ dimensional ball, $B^n$.

Figure 1. Medial axis transform for 2D and 3D shapes (a)(b)(c); (d) and (e) highlight how small changes in the polygon are reflected on the skeleton.

Differential topology suggests another approach to shape description which mainly relates to Morse theory, [17][24]. Given an object surface $S$ and a Morse function $f : S \rightarrow \mathbb{R}$, Morse theory states that the shape of the pair $(S, f)$ is represented by the evolution of the homology groups of the level sets:
$S' = f^{-1}((\infty, x])$ as $x$ varies on $\mathbb{R}$, [22]. Since homology and homotopy groups code shape properties as the number of connected components, holes and cavities in the object, it follows that a finite collection of level sets is sufficient to fully describe the surface shape and it is more satisfactory than the simple knowledge of the global homology, [22]. Focusing on the level set evolution, we obtain a discrete description which effectively represents the shape of $S$ and can be encoded in a topological graph, as formalized by the following definition, [15][30].

**Definition.** Let $f : S \rightarrow \mathbb{R}$ be a real valued function on a compact manifold $S$. The Reeb Graph $G$ of $S$ with respect to $f$ is the quotient space of $S \times \mathbb{R}$ defined by the equivalence relation $\sim$, given by:

$$(P, f(P)) \sim (Q, f(Q)), P, Q \in S \quad \text{iff} \quad f(P) = f(Q) \quad \text{and} \quad P, Q \in \text{the same connected components of} \quad f^{-1}(f(P)).$$

Mathematically, the Reeb graph is defined as the quotient space of $S$ with respect to the value of the function $f$. Moreover, the critical points of $f$, that is the points of $S$ where the gradient of $f$ vanishes, correspond to topological changes of the Reeb graph of $S$, determining its homology groups and how cells, which correspond to critical points, glue in the graph (see Figure 2).  

![Figure 2. The Reeb graph of a complex object with nine holes when $f$ is the height function depicted.](image)

Since the properties of $S$ and $f$ determine those of $G$, the quotient function $f$ has to be chosen in order to extract characteristics which fully describe the object with respect to the application needs. From a topological point of view, the continuity of $f$ ensures that two homeomorphic manifolds are mapped into homeomorphic graphs thus guaranteeing their identification. Even if previous considerations enable to fully identify topological properties of $S$ through $G$, different application fields, such as matching, compression, etc., require to select, among different but equivalent representations of $S$, that more suitable for the algorithm which uses it as input. For instance, matching requires an input graph with a minimal number of redundant leaves in order to avoid a preprocessing for pruning. In addition, the Reeb graph requires the storage of an amount of geometric information for reconstruction tasks, [6]. Therefore, according to the application domain, the choice of the most appropriate function $f$ varies and should enable the extraction of some geometric information on $S$, such as section length, area, etc.

The family of continuous or Morse functions is a natural set for identifying $f$, even if the choice of a candidate faces with computational constraints. In the next section we present an overview of possible choices of $f$ for coding triangular meshes without boundary, with special emphasis on their properties and differences.

## 4. Overview of topological graphs and their properties

In this section an overview of possible choices of $f$ for building the Reeb graph of a closed manifold surface of arbitrary genus represented through a triangle mesh is presented. However, for preferring a particular function $f$ with respect to another one, a constraint on its choice is a good compromise between calculation, invariance and description effectiveness.

First of all, according to the definition context, the mapping functions are grouped in two main classes: maps which are mathematically well known and are applied to the discrete context, and those directly defined on the mesh. Then, we emphasize the main properties and drawbacks of the resulting Reeb graphs, in particular, highlighting the most suitable applications.

- **Reeb Graph with respect to the height function**

A standard choice of $f$ is the height function in the three-dimensional space (see Figure 6 in the colour plate), which has extensively been studied in [3][7][32]. The equivalence classes induced by the height function correspond to the intersection of the mesh with a set of planes that are orthogonal to a given direction. In this case, the critical points correspond to peaks, pits and saddles of the object; the generated graph very intuitively resembles the skeleton of the object silhouette but it depends on the
chosen direction of the height function, so that different orientations may produce different results.

Although the methods presented in [32] and in [3] have different input requirements and computational costs, the properties of the extracted graph are comparable; in particular, Morse theory guarantees that, with a suitable object slicing, the number of cycles in the graph corresponds to the number of holes of $S$ but does not answer to the problem of calculating an optimal slicing of the model. As shown in [3], the choice of the slicing thickness is critical and the density of slices determines the scale of the shape features detected. A possible solution is to slice the mesh in correspondence of each value of $f$, as shown in [11], but this approach involves the computation of too many slices. The hierarchical approach to the Reeb graph extraction proposed in [3] is obtained by considering the maximum and the minimum value of $f$, $\min_{\text{max}} f$, and halving that interval until the distance among the contour levels is less than a given threshold $T$. Furthermore, the topological correctness of the graph is guaranteed by an adaptive slicing process that refines the thickness of contour levels in correspondence of holes.

The main property of the Reeb graph calculated with respect to such a $f$ is the independence from translations and uniform scaling. Moreover, the evaluation of the function $f$ is immediate and the graph is a very powerful tool for understanding the shape in contexts where surfaces have a naturally privileged direction, as terrain models. [7][11][33]. Furthermore, starting from the sections that are the boundary components of the critical regions of $f$ and following the connectivity relationship coded in the Reeb graph, a topologically consistent framework for surface reconstruction has been proposed in [6]. However, the dependence on the height function on rotations, makes this graph unsuitable for matching and classifying 3D shapes.

- **Reeb Graph with respect to the distance from the barycentre**

  Differential topology suggests another class of Morse maps: the distance functions of the surface points from a given point $p$ of the Euclidean space. Such a point could belong to the mesh or not, even though a reasonable choice is the barycentre of the object which is easily calculated and, due to its linear dependence on all the vertices, stable to small perturbations. Then, the evaluation of the function on the mesh vertices is straightforward: in fact, at each vertex $v$ of the mesh, the Euclidean distance between $v$ and $p$ is associated. According to the criteria proposed in [3], the isocontours of the distance map on $S$ can be detected by interpolating the values of $f$ (for example, see Figure 7(a), in the colour plate). The contour levels, which are computed for the height function, induce a characterization of the regions of the mesh, which is done by comparing the value and the number of each boundary components (see Figure 7(b)-(c)). Depending on this choice of $f$, the interpretation of areas of maximum and minimum differs from those obtained with the height function; in fact, the isocontours correspond to the intersection of the mesh with a collection of spheres centred in the barycentre and with different radii. In other words, it is possible to recognize a set of protrusions and hollows of the mesh with respect to the barycentre by analysing the maxima and the minima of the function $f$ that have only one boundary component. An example of the different behaviour of minima is shown in Figure 7(b): the minima corresponding to the palm and the back of the hand represent two concavities while minima on the ring and little finger locate two surface protrusions.

  Because the barycentre and the sphere/mesh intersection are independent on translation, rotation and uniform scaling of the object, these properties are reflected on the resulting Reeb graph. An example of Reeb graph calculated with respect to the barycentre distance is depicted in Figure 7(d).

- **Reeb Graph with respect to the geodesic distance**

  A different mapping function has been defined by Hilaga et al. [19], where the notion of an integral geodesic distance has been introduced for matching purposes. In particular, for each vertex on a triangulation $S$, the value of the function $f$ is given by:

  \[
  f(v) = \int_{p \in S} g(v, p) \, ds
  \]

  where $g(v,p)$ represents the geodesic distance between $v$ and $p$, when $p$ varies on $S$. This function is not invariant to scaling of the object and it is replaced by its normal representation defined as

  \[
  f(v) = \sum_{b_i \in B} g(v, b_i) \cdot \text{area}(b_i),
  \]

  where $\{v_i\} = \{v_1, ..., v_k\}$ are the base vertices for the Dijkstra’s algorithm which are scattered almost equally on the surface and $\text{area}(b_i)$ is the area of the neighbourhood of $b_i$. The resulting Reeb graph is
theoretically invariant with respect to rotation, translation and uniform scaling.

Even if the surface curvature and the geodesic distance are good shape estimators, their dependence on second order derivatives makes them numerically unstable, preventing their direct use for the graph definition. Due to the time complexity of the exact evaluation of the function $f$, an approximation based on the Dijkstra's algorithm has been proposed. Unfortunately, this choice does not guarantee the absolute independence of its values from the object orientation and it is computationally expensive. Examples of evaluation of the function $f$ are depicted in Figure 3 and the graph structure is shown in Figure 8 of the colour plate.

![Figure 3. Isocontours of the function in [19].](image)

The previous choices of $f$ highlight the limits coming from the exact evaluation of $f$ on the mesh vertices: in fact, an approximation of the mapping function or an interpolation of vertex values is introduced during the contour level calculation. For this reason, recent efforts define the map directly on the mesh such that the produced graph is significant and invariant to affine transformations; examples have been proposed in [20][21][25][36].

- **Centerlines based on discrete geodesic distance from a source point**
  
  For representing three dimensional polyhedral objects a first approach which deals with the construction of centerlines from unorganised point sets has been presented in [21], and later developed for polyhedral objects in [20]. With reference to [20], a skeleton-like structure, available for triangular meshes homeomorphic to a sphere, is proposed, which is essentially a tree made of the "average points" associated with the connected components of the level sets of a given function; in particular, the geodesic distance from a source point is chosen, as shown in Figure 9 in the colour plate. To automatically select the source point an heuristic is used, which seems to work well on elongated tubular shapes. In this case, skeletal lines obtained with different source points are very similar and the resulting skeleton is invariant under rotation, translation and scaling. Anyway, the choice of only one source point determines a privileged "slicing direction", which can lead to the loss of some features if the object is not tubular shaped (like the horse ears in Figure 4(b)).

![Figure 4. Isolevels (a) and the centerline (b) of the horse as computed in [21].](image)

Almost an extension of the previous approach to non-zero genus surfaces has been presented by Wood et al. [36]: there, the graph is implicitly stored for generating high quality semi-regular multi-resolution meshes from distance volumes. Also in this case, the object topology is achieved by considering a wavefront-like propagation from a seed point, [4] (see Figure 5). The calculation of the isosurfaces is obtained by applying the Dijkstra's algorithm; this makes this approach unavailable for non-uniform scaling.

![Figure 5. The wave-front propagation in [36].](image)

- **Skeleton based on topological distance from curvature extrema**
  
  The strategy proposed in [25] extracts the skeleton of a surface represented by a simplicial complex and combines differential and computational topology techniques. In this context, a multi-resolution
curvature evaluation, [13] is introduced to locate seed points which are sequentially linked by using the natural topological distance on the simplicial complex (see Figure 10(a),(b) in the colour plate). More precisely, once computed the approximated Gaussian curvature for the mesh vertices, for each high curvature region \( R_i \), \( i = 1 \ldots n \), a representative vertex \( p_i \) is selected. Starting at the same time from all the representative vertices, rings made of vertices of increasing neighbourhoods are computed in parallel until the whole surface is covered (see Figure 10(c)), in a way similar to the wave-traversal technique [4]. Rings growing from different seed points will collide and join where two distinct protrusions depart, thus identifying a branching zone; self-intersecting rings can appear expanding near handles and through holes. A skeleton is drawn according to the ring expansion: terminal nodes are identified by the representative vertices, while union or split of topological rings give branching nodes. Arcs are drawn joining the centre of mass of all the rings (see Figure 10(d)).

The function \( f_n(x) = \min\{k : x \in k - \text{neighborhood}\} \) defines the topological distance of \( x \) from \( p_i \), and can be extended to a finite set of vertices \( \{p_1, \ldots, p_n\} \) as

\[
f(x) = \min_{i=1}^{n} f(p_i(x)), \quad \forall x \in S, \quad \text{i.e.} \quad f \text{ assigns to } x \text{ its minimal topological distance with respect to more than one vertex. Given } S \text{ and } f \text{ it follows that } G = S^f \text{ is a Reeb graph with respect to such a function } f.
\]

The complexity of the proposed graph, in terms of number of nodes and branches, depends on the shape of the input object and on the number of seed points which have been selected using the curvature estimation criterion. This graph is affine-invariant (translation, rotation, scaling) because the chosen function \( f \) does not rely either on a local coordinate system or on surface embeddings as it happens, for example, using the height function. On the other hand, if the curvature evaluation process does not recognize at least one feature region, e.g. surfaces with constant curvature value as spheres, this approach is not meaningful for extracting a description of the shape; on the contrary, the height function always guarantees to get a result. Finally, experimental results have shown that this framework works on shapes of arbitrary genus.

5. Discussion and Conclusions

In the previous section, several topological graphs and their basic properties have been described. As already mentioned, the choice of the best suited shape descriptor depends on the application domain. We are now concerned with a discussion about the pros and cons of the described graphs in relation with different tasks; in the following \( n \) is assumed to be the number of vertices of the input mesh.

Traditionally, the MAT is the most popular skeleton representation because it is available both for reconstruction and animation purposes. However, its dependence on small perturbations and noise makes it not very suitable for recognition and matching contexts. Moreover, due to the large computational cost of 3D algorithms \( O(n^4 \log n) \) in the worst case, [31], it is more used in image contexts than in the polyhedral ones.

The Reeb graph with respect to the height function intuitively resembles the skeleton of the input shape and is practically easy to extract: the object is sliced with a set of parallel planes coding the evolution of the intersection sections. The computational cost of the graph construction is \( O((n-k) \log(n+k)) \) where \( k \) is the number of vertices inserted in the mesh during the slicing phase, as shown in [3]. Moreover, the slicing can be selectively refined giving a multi-resolution description of the shape. The affordable computational time required for the graph extraction and the chance to attach geometric information to arcs and nodes (for instance, the critical section of the objects, at which topological changes occur) make it feasible for compression and reconstruction purposes, as done in [6]. An approximated shape can be recovered from the critical sections, and the adjacency relation among them, necessary in branching situations, are given by the graph structure. However, the dependence on the extracted graph on the direction of the height function makes it ineffective for matching applications: the same object may have completely different graphs depending on its position in space, thus invalidating even the identity property. It remains that in particular cases where a preferential object orientation exists, such as in digital terrain modelling, the Reeb Graph with respect to the height function is effective for matching applications too.

The distance from the barycentre of the input data set induces a Reeb Graph which is invariant to the
object position in space, resulting suitable for the similarity analysis. Reconstructing an approximated shape from the achieved graph would be quite complex, since isocontours are not planar. In this description, nodes correspond to protrusions and hollows with respect to the center of mass; the vertex classification based on the distribution of the distance function on the object surface is similar to that shown by the graph based on geodesic distance in those cases where the center of mass corresponds to the center of the surface in the geodesic sense. The behaviour of the two representations has been taken into account in similarity analysis: for instance, the geodesic distance distribution on a frog does not change if the legs are stretched rather than curled up, since the geodesic distance from the body does not change, while the Euclidean distance from the center of mass does. Therefore, if the intent is to distinguish between different poses of the same object, the Reeb Graph with respect to the distance from the barycentre should be preferred. Furthermore, the computational cost for evaluating the function $f$ is $O(n)$ for the barycentre distance and $O(h \log n)$ (due to Dijkstra’s algorithm) for the function in [19], while the cost of the graph extraction is $O(n+k \log(n+k))$ and $O(n+k)$, respectively, where $k$ is the number of vertices inserted in the mesh. Moreover, the representation proposed by [19] has been effectively applied for matching purposes, but seems not suitable for reconstruction.

The centerline extraction based on geodesic distance, conversely, is strongly affected by the need of a source point, from which an approximated geodesic distance to each vertex is computed. To make a search key of 3D shapes, the source point must be determined automatically and stably, thus resulting in a difficult problem. For example, a small change in the shape may locate a different source point, creating an obstacle for the construction of a stable graph. Another problem of this representation is that important features may be lost when the input object is not strictly tubular: the choice of a starting point determines also a preferred slicing direction, so that protrusions perpendicular to that direction are not captured. This is actually the same problem affecting the Reeb Graph with respect to the height function. The possible loss of important features also makes this graph not suitable for reconstruction too, while the limitation of treating only zero genus solids has been overcome in [36]. In spite of these limitations, it is intuitive and useful for finding centerlines of tubular shapes, in applications like finding a central path in blood vessels and human organs, and it is computationally cheap ($O(h \log n)$).

The curvature-based skeleton is generated starting from the object protrusions, which are first determined through a robust to noise multi-scale curvature evaluation on the surface. Details are discarded since the curvature evaluation is performed at a wide range of scales, and relevant features at each scale can be chosen through a query or collected together. The curvature evaluation is time consuming but, once chosen the protrusion features as seed points, the expansion of topological circles on the surface and the growth of graph edges take linear time. These properties make this graph available both for matching and object manipulation and, by adding geometric information on nodes and along arcs, for reconstruction purposes.

In the Table 1 a summary of main application tasks of the skeletons previously described is presented. With reference to the table, the symbol $\oplus$ means “suitable”, while $\otimes$ stands for “unsuitable”; where $\oplus$ appears it means theoretically possible but not easy to apply. Computational costs are shown in the last three columns. Note that the extraction of the medial axis transformation depends on the number of line elements, $n$, the number of steps taken along each seam, $s$, and the number of boundary entities $n$ of the MAT (details can be found in [31]). Furthermore, the cost of the evaluation of the function $f$ is given both in the average and in the worst case. For example, the distinction between average and worst case is important in the method presented in [19], where during the function evaluation a number $b$ of triangle bases is introduced, (in practice $b$ is less than 150), and the function is calculated starting from them; in this way, the computational complexity is considerably improved. Finally, the average cost of the curvature evaluation of the method in [13] depends on two entities: the number of mesh vertices $n$, and $k_{max}$ which is the size of maximum neighbourhood of each vertex used to approximate the curvature.

In this paper we have presented the effectiveness of structural graphs, that is topological structures enriched with proper geometric information, in different fields of computer graphics. As previously
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Table 1 Summary of the main skeleton properties. Rg=recognition, Rc=recognition, Morph= morphing, Match= matching and A= animation.

Acknowledgements

The figures 3, 4(b) and 5 are extracted from [19], [21], and [36], respectively.

This work has been partially supported by the National Project "MACROGeo: Metodi Algoritmici e Computazionali per Rappresentazione di Oggetti Geometrici", FIRB grant.

The authors would like to thank all the people of the Computer Graphics Group at IMATI-CNR, in particular Bianca Falcidieno and Michela Spagnuolo for their useful suggestions.

References


Figure 6. (a) Topology changes with respect to the direction $f$, (b) Reeb graph with a minimal number of surface sections.

Figure 7. The characterization process of the mesh with respect to the distance from the barycentre, (a), (b), (c), and its Reeb graph, (d).

Figure 8. The graph representation with respect to the geodesic distance in [19].

Figure 9. A graph-like representation of the centerline proposed in [20].

Figure 10. (a) Vertex classification based on Gaussian curvature (b) high curvature regions are depicted in red; (c) topological rings expanded from the centres of high curvature regions (d) obtained graph.