Differential topology methods for shape description*

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Differential topology, and specifically Morse theory, provides a suitable setting for formalizing and solving several problems related to shape analysis. In this field, we discuss how a shape can be analyzed according to the properties of a real function defined on it (e.g., harmonic fields or laplacian eigenfunctions), and how these properties can be stored in compact and informative descriptors. We refer to Reeb graphs, that encode the configuration of level sets and critical points of the function.

Introduction. The analysis and description of 3D digital shapes are key-issues in the current scientific scenario, playing a role in industrial design, bioinformatics, compuer-aided diagnosis, physical and engineering simulation, to cite a few. Mathematics can support these tasks by providing a set of tools (e.g., homotopy or homology) that can be reread from a computational viewpoint and used to formalize and solve shape analysis problems. A possible source of inspiration is classical Morse theory: the fundamental idea is to combine the topological exploration of a shape with quantitative measurements of geometrical properties, the latter provided by a real function defined on the shape. The added value is that different functions can be used according to the properties and invariants that one wishes to capture: real functions play indeed the role of a lens through which we look at the shape. The approach we discuss works in two steps: first, one or more real functions are evaluated on the shape; then, a shape descriptor is defined to quantify in a compact structure the information conveyed by these functions. After discussing different choices of real functions, we describe how to synthesize their behaviors by coding the configuration of the associated level-sets in a Reeb graph.

Describing shapes via real functions. A variety of different functions have been proposed to analyze shapes, in several application fields which range from surface remeshing and parameterization to 3D shape matching and retrieval. If any *a-priori* information is available on the input shape, this can be used to select the functions which are best suited to identify specific shape features. By *shape feature* we mean a property characterizing the shape and that is relevant in a given context (e.g., protrusions or holes); then, the choice of the function drives the analysis by constraining the description to interpolate such properties. The *height* function is among the most intuitive and simple choices for analysing the shape of a surface, but its drawback is the dependence on the direction considered. A more elaborate characterization is provided by the *elevation* [1] function, which derives from the height function but aims at a rotation invariant analysis. Roughly speaking, the elevation measures how much a point is relevant in its normal direction with respect to its neighbourhood and it has been used to detect cavities and protrusions in docking studies. The Euclidean or geodesic distance of mesh vertices from selected feature points [2, 3], or the average of all geodesic distances among the vertices [4], can be applied to guarantee a description that retains or forgets the spatial pose. Also *curvature-based analysis* have been frequently used to characterize the shape of 3D surfaces. Since the curvature-based analysis is rather sensible to noise, small features, and the quality of the shape discretization in terms of sampling density and tiny triangles, a more robust approximation of the curvature values is achieved either using variations of the curvature evaluation function [5], polynomial surface fitting, or a multi-scale curvature evaluation where details are discarded [6].

In case the set of input shapes does not exhibit a uniform structure, *harmonic* [7] and *Laplacian-based* functions [8] may provide a new and powerful set of descriptors for shape analysis, as they are intrinsically defined by the Laplace-Beltrami operator Δ. In [7], a harmonic function f is calculated by solving the equation $\Delta f = 0$ subject to the Dirichlet boundary conditions $\mathcal{B} := \{f(\mathbf{p}_i) = a_i,\ i \in \mathcal{I}\}$, $\mathcal{I} \subseteq \{1, \ldots, n\}$. For piecewise linear functions on triangulated surfaces, the discrete Laplacian operator is defined as $\Delta f(\mathbf{p}_i) = \sum_{j \in N(i)} w_{ij}\,[f(\mathbf{p}_j) - f(\mathbf{p}_i)]$, where $N(i)$ is the set of vertices adjacent to the vertex i and w_{ij} the weight associated with the directed edge (i, j). As coefficients w_{ij} we can select the *mean-value* or the *cotangent weights*, which approximate harmonic maps or minimize the Dirichlet energy respectively. Then, computing f requires to solve a sparse linear system whose coefficient matrix is related to the *Laplacian matrix* L which discretizes Δ. Another choice is to consider the scalar function corresponding to the eigenvector $\mathbf{x}_i$ of L related to the eigenvalue λ_i; in this case, $f_i := \sqrt{\lambda_i}\mathbf{x}_i$ with $L\mathbf{x}_i = \lambda_i \mathbf{x}_i$, $i = 1 \ldots, n-1$. The functions related to the smallest eigenvalues are generally smooth, with a low number of critical points; they also show slow variations, while those related to higher eigenvalues show rapid oscillations.

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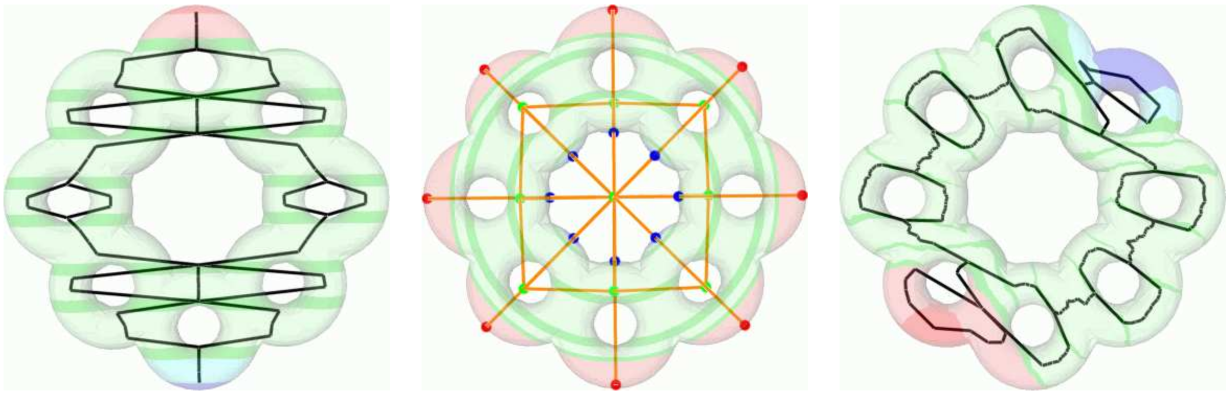

Fig. 1 Reeb graph of a 2-dimensional manifold studied with respect to the height function (left), the distance from the center of mass (middle) and the second eigenfunction of the Laplacian matrix (right).

Shape descriptors. The functions used to analyze the shape can be directly associated to a corresponding signature, or shape descriptor. This association can be exclusive, meaning that a specific function has to be used to produce a given signature, or the descriptor can be *parametric* with respect to the choice of the function. The added value of the latter approach relies on the possibility of adopting different functions, according to the properties and invariants that one wishes to analyze. In this class of descriptors [7, 9, 10], the Reeb graph is probably the most popular. It dates back to 1946, when the French mathematician George Reeb [11] gave its definition for a Morse function f defined on a smooth manifold $\mathcal{M}$, in terms of the quotient space defined by the equivalence relation that identifies the points belonging to the same connected component of level sets of f:

> *Let $\mathcal{M}$ be a compact manifold of dimension n and f a simple Morse function defined on $\mathcal{M}$, and let us define the equivalence relation "$\sim$" as $(\mathbf{p}, f(\mathbf{p})) \sim (\mathbf{q}, f(\mathbf{q}))$ if and only if $f(\mathbf{p}) = f(\mathbf{q})$ and $\mathbf{p}, \mathbf{q}$ are in the same connected component of $f^{-1}(f(\mathbf{p}))$. The quotient space on $\mathcal{M} \times \mathbb{R}$ induced by "$\sim$" is a finite and connected simplicial complex K of dimension 1, such that the counter-image of each vertex Δ_i^0 of K is a singular connected component of the level sets of f, and the counter-image of the interior of each simplex Δ_j^1 is homeomorphic to the topological product of one connected component of the level sets by $\mathbb{R}$ [11].*

The quotient space defined by Reeb is what is currently called Reeb graph. Since its introduction in Computer Graphics by Shinagawa et al. [12] in 1991, Reeb graphs have been used to solve different problems related to shape matching, morphing and coding. The Reeb graph acts as a tool for studying shapes through the evolution and the arrangement of the level sets of a real function defined over the shape. It is able to convey both geometrical and topological information, since the topological analysis is driven by the properties expressed by f. Its parametric nature with respect to f is shown in Figure 1, where the Reeb graph of a closed surface with respect to different functions are depicted. Notice how different functions can give insights on the shape from a different perspective.

Concluding remarks and future perspectives. Shape descriptors relying on the use of a real function defined on the shape, and particularly descriptors based on differential topology and Morse theory, play a fundamental role in the field of shape analysis. In this area, an interesting issue concerns the way to use concurrently multiple functions. In this scenario, we foresee the definition of a new generation of descriptors, that would allow an efficient analysis of complex and high-dimensional data.

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