A TOPOLOGY-BASED APPROACH TO SHAPE MODELING

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Abstract:

This paper deals with the mathematical background and application understanding that is behind the representation and generation of shapes in computer graphics. In particular, special emphasis is given on issues related to the definition of abstraction tools for deriving high-level descriptions of complex shape models. Among the wide range of shape descriptors, topological graph-like representations not only give a powerful and synthetic sketch of the object, but also capture its inner structure, that is how features connect together to give the overall shape. This aspect make them useful to describe complex 3D objects in various applications as modelling, morphing, matching and recognition.

Keywords: shape understanding, Reeb graph, skeleton.

1 Introduction

A possible approach to the problem of shape classification and understating is to represent shape properties of a given object through shape descriptors which are useful to detect those properties, which are invariant to position, orientation, local noise and other distortions. Among all, descriptors based on geometry and topology seem to be suitable for dealing with the definition of basic models for representing and generating shapes. Computational topology has been recently proposed as a branch of Computer Graphics, which aims at solving problems related to topological issues, without forgetting the feasibility and the computational complexity of the problem. For instance, topology can be successfully applied as a basis for the compression of dense meshes where the knowledge of the adjacency relationship among facets is locally exploited to decrease the amount of stored information [5]. Furthermore, this theory can be used to locate the main features of the object discarding details in the mesh without losing the overall appearance [4]. Finally, content-based search in object databases and shape-based processing are conveying an increasing attention on the formalization of shape as a combination of geometry and semantics, in order to define tools for assessing the similarity among different models [13]. Shape interpretation is especially relevant for the perception of complex forms, where the ability of varying the level of descriptive abstraction is the key for recognizing and classifying highly complex shapes through a multi-resolution framework which abstracts the object at different levels of detail. From a mathematical point of view, Morse theory, homotopy and homology appear to be appropriated tools to deal with topological questions in computer graphics applications [8,9]. The reminder of the paper is organized as follows: in Section 2, the theoretical background on Morse theory is briefly summarized considering all those elements which have a direct application for the extraction of the Reeb graph on triangular mesh as described in Section 3. Conclusions and future improvements are proposed in the last section.

2 Theoretical background on Morse Theory

Our approach to shape description analyses any data that can be modelled as a surface and it is mainly related to Morse theory. Given an object surface *S* and a Morse function $f : S \to \mathbb{R}$, Morse theory states that the shape of the pair (S, f) is represented by the evolution of the homology groups of the sets $S^x := f^{-1}((-\infty, x])$ as *x* varies on $\mathbb{R}$. Moreover, the critical points of *f* give information about the global topology of *S* and determine its homology groups. Since homology groups codify shape properties as the number of connected components, holes and cavities in the object, it follows that a finite collection of level sets is sufficient to fully describe the surface shape and is more complete than simply knowing the global homology. Focusing on the evolution of S^x, we obtain a discrete description which effectively represents the shape of *S* and can be encoded in a topological graph, as formalized in the following, [11].

Definition. Let $f : S \to \mathbb{R}$ be a real valued function on a compact manifold *S*. The *Reeb Graph G of S with respect to f* is the quotient space of $S \times \mathbb{R}$ defined by the equivalence relation ~, given by $(P, f(P)) \sim (Q, f(Q)), P, Q \in S$ iff $f(P) = f(Q)$ and *P*, *Q* are in the same connected component of $f^{-1}(f(P))$.

The Reeb graph, defined as the quotient space of *S* wrt the function *f*, effectively codes the topological evolution of the surface contour levels $f^{-1}(f(P))$ at each point *P* on *S* by considering how cells corresponding to critical points glue together (see Figure 1).

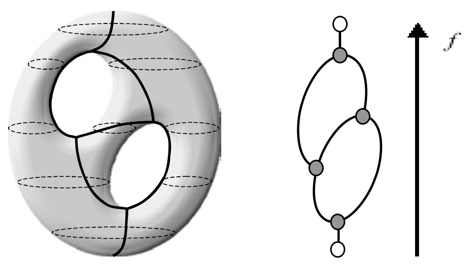

Fig. 1 : *The equivalence classes and the Reeb graph of a bi-torus wrt a specific height function f.*

Since the properties of *S* and *f* determine that of *G*, the quotient function *f* has to be chosen in order to extract characteristics which fully describe the object with respect to the application needs. From a topological point of view, the continuity of *f* ensures that two homeomorphic manifolds are mapped into homeomorphic graphs thus guaranteeing their identification. Even if previous considerations enable to fully identify topological properties of *S* through *G*, different application fields, such as matching, compression, etc., require to select, among different but equivalent representations of *S*, that more suitable for the application context. For instance, matching requires an input graph with a minimal number of redundant leaves in order to avoid a pre-processing for pruning. These needs result in searching for a function *f* which produces an affine-invariant graph of 3D shapes [10], which enables to extract geometric information on *S* such as section length, area, etc. Usually, the function *f* is chosen in the family of Morse functions, always taking into account the trade off between calculation, invariance and description effectiveness. Possible choices are described in the following section.

3 Graph representation for triangular meshes

A standard choice of *f* is the height function in the three-dimensional space (see Figure 2) which has been extensively studied in [2,3,12]. In this case, the critical points correspond to peaks, pits and saddles of the object and the generated graph very intuitively resembles the skeleton of the object; unfortunately, it depends on the chosen direction of the height function, so that a different orientation may produce a different graph.

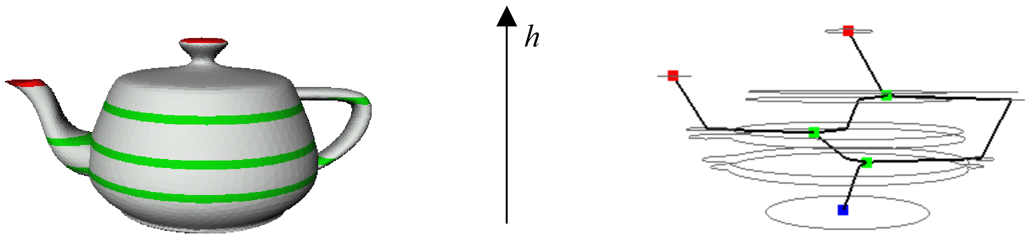

Fig. 2**:** *The Reeb graph wrt the direction h with a minimal number of surface sections (right); on the surface we highlight those parts of the surface where topology changes (left).*

For this reason, recent efforts are focused on the definition of a map such that its related graph is invariant to affine transformations; examples have been proposed in [7,14,15].
Surface curvature and geodesic distance seem to be good shape estimators; unfortunately, their dependency on second order derivatives makes them numerically unstable preventing their direct use for the graph definition. Instead we have proposed a mixed strategy, which extracts the skeleton of a surface represented by a simplicial complex, combining differential and computational topology techniques. In this context, a multi-resolution curvature evaluation [6] is introduced to locate seed points which are sequentially linked by using the natural topological distance on the simplicial complex (see Figure 3a). More precisely, once computed the approximated Gaussian curvature for the mesh vertices, for each high curvature region R_i, a *representative vertex* p_i is selected. Starting at the same time from all the representative vertices, rings made of vertices of increasing neighbourhoods are computed in parallel until the whole surface is covered, in a way similar to the wave-traversal technique [1]. Rings growing from different seed points will collide and join where two distinct protrusions depart, thus identifying a branching zone; self-intersecting rings can appear

expanding near handles and through holes. A skeleton is drawn according to the ring expansion: *terminal nodes* are identified by the *representative vertices*, while union or split of topological rings give *branching nodes*. Arcs are drawn joining the centre of mass of all the rings (see Figure 3b).

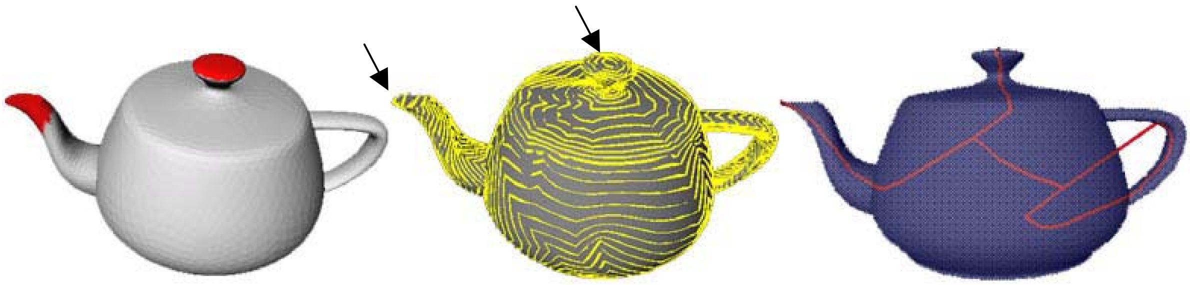

Fig. 3: *(a) High curvature regions are depicted in red; (b) expansion of the neighbourhood rings from the representative vertices; (c) obtained graph.*

We are now concerned with the formalization of the graph construction for triangle meshes as quotient space; its generalization to manifolds is analyzed in [10]. Selected a point p_i, we introduce the function $f_{p_i}(x) := \min\{k : x \in k-neighborhood\}$ which defines the topological distance of x from p_i. We can extend the previous function to a finite set of vertices $\{p_1, \ldots, p_n\}$ as $f(x) := \min_{k=1,\ldots,n}\{f_{p_k}(x)\}, \forall x \in S$, i.e. f assigns to x its minimal topological distance with respect to more than one vertex. Starting from S and f, we are able to construct the relation ~ as:

$$p, q \in S, p \sim q \Leftrightarrow f^{-1}(f(p)) \cap f^{-1}(f(q)) \neq \emptyset.$$

From the previous definition it follows that $G=S/\sim$ and it is a Reeb graph wrt such a function f. The complexity of the proposed graph, in terms of number of nodes and branches, depends on the input shape and on the number of seed points which have been selected using the curvature estimation criterion. This graph is affine-invariant (translation, rotation, scaling) because the chosen function f does not rely either on a local coordinate system or on surface embeddings as it happens, for example, using the height function. On the other hand, if the curvature evaluation process does not recognize at least one feature region, e.g. surfaces with constant curvature value as spheres, our approach is not meaningful for extracting a description of the shape; on the contrary, the height function always guarantees to get a result. Finally, we observe that the proposed framework works on shapes of arbitrary genus.

3 Conclusions and future work

The proposed ideas are starting steps for reconstructing a complete framework available for shape abstraction, classification (i.e. feature detection), comparison (i.e. matching) and editing (i.e. morphing). As previously highlighted, the choice of different shape descriptors can be done according to the application needs allowing certain degree of freedom; indeed, the use of topological graphs provides flexible tools for the abstraction of complex shapes with arbitrary topology. Currently, we are investigating further improvements in the choice of the representative function, which should also guarantee a better differentiability properties.

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