Analysis and Comparison of Real Functions on Triangulated Surfaces

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Abstract. The study of geometric and topological properties of a triangulated surface $\mathcal{M}$ is ubiquitous in geometry and shape processing. In these contexts, different techniques in research areas such as shape abstraction, comparison, and parameterization are based on the study of local properties of a mapping function $f : \mathcal{M} \rightarrow \mathbb{R}$ and that correspond to specific properties of the input shape. Therefore, it becomes interesting to analyze if two or more functions are "*independent*", that is, how much and where the measured properties differ. This paper proposes a set of descriptors to analyze, compare, and model a family of functions $(\mathcal{M}, f)$, when f varies.

§1. Introduction

Differential topology, and specifically Morse theory, provides a mathematical setting suitable for several problems related to shape analysis, abstraction, and comparison. The intuition behind Morse theory is that of combining the topological exploration of a shape $\mathcal{M}$ with quantitative measurements of its geometric properties provided by a mapping function $f : \mathcal{M} \rightarrow \mathbb{R}$ defined on $\mathcal{M}$ [7]. The added value to shape analysis of approaches based on Morse theory relies on the possibility of adopting different functions as descriptors, according to the properties and invariants that one wishes to analyze. As f varies on the input shape, several properties of $\mathcal{M}$, such as critical points and iso-contours, are measured through f and provide insights on $\mathcal{M}$ from different perspectives. Therefore, the flexibility of the choice of f faces the problem of defining a qualitative comparison of the properties measured by two or more functions.

Solutions to this problem are provided by (a) the *correlation coefficient*, which statistically measures how much the values of the functions are

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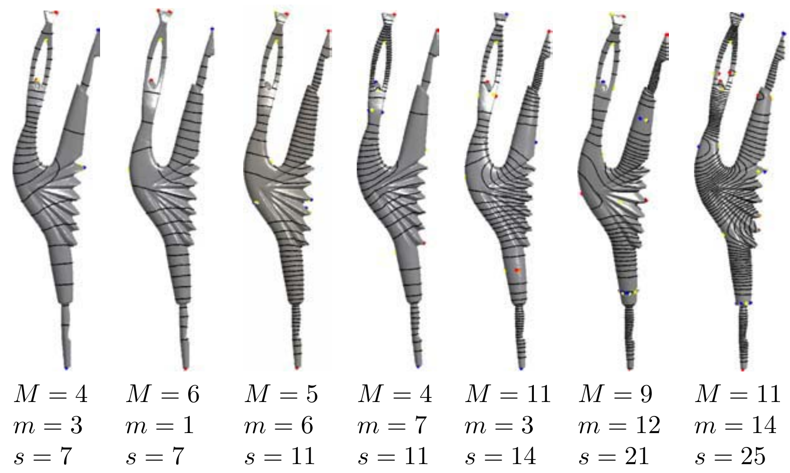

Fig. 1. Iso-contours and critical points (M maxima, m minima, and s saddles) of the first seven Laplacian eigenfunctions [3].

independent or not; (b) the *Earth mover's distance* [9], which evaluates the amount of work needed to transform a function into another. Both methods can be used to compare an arbitrary number of functions and they provide global measures without describing local differences. On the contrary, [4] proposes both a local and global comparison measure, which are based on the k-form of a collection of k functions. In the case of two functions f and g defined on a surface, the local measure at $\mathbf{p} \in \mathcal{M}$ is the value of the cross-product of the gradients of f, g at $\mathbf{p}$ and the global descriptor is defined as an averaged sum of these values.

In our approach, we deal with closed and manifold triangle meshes. In this case, we consider two functions as "*similar*" if they have an analogous behavior on the same regions of $\mathcal{M}$ and we estimate this similarity by studying the differences of the level sets associated to both f and g. Our assumption is that differences and analogies of the level sets of these functions indicate differences and similarities of their behavior on $\mathcal{M}$ (see Fig. 1). More precisely, to evaluate the similarity of the couples $(\mathcal{M}, f)$ and $(\mathcal{M}, g)$, we define a new functional $\mathcal{I}(f, g) : \mathcal{M} \rightarrow \mathbb{R}$ that measures the angle variation of their gradient fields (see Section 2). As original contribution with respect to the previous work, we provide a direct relation between the critical points of f, g, and $\mathcal{I}(f, g)$. Then, we generalize this problem to an arbitrary number of functions defined on $\mathcal{M}$. Given $(\mathcal{M}, f)$, we also use the similarity functional to calculate a new $g : \mathcal{M} \rightarrow \mathbb{R}$ that

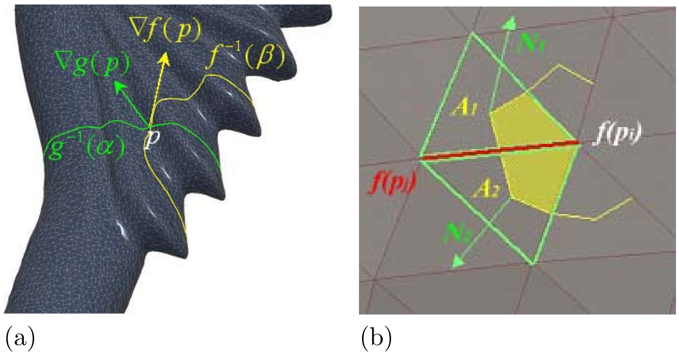

Fig. 2. (a) Iso-contours of two functions f and g that intersect at $\mathbf{p}$. (b) Discretization of the gradient field of f at $\mathbf{p}_i$ with respect to its $1-$star.

is "*orthogonal*" to $(\mathcal{M}, f)$ (i.e., the most dissimilar) with respect to $\mathcal{I}$ and we propose an efficient algorithm for its computation (see Section 3). In this way, we provide a locally orthogonal coordinate system on the surface, a property vital for various applications, like a consistent computation of geodesic distances on the surface. Finally, in Section 4 concluding remarks, dealing with possible forthcoming areas of development, are proposed.

§2. Comparison of real functions: continuous case

Let $\mathcal{M} \subset \mathbb{R}^3$ be a 2-manifold and $f : \mathcal{M} \to \mathbb{R}$ be a function of class C^k, $k \geq 1$; then, the *gradient* of f is defined as $\nabla f := (\partial_{x_1} f, \partial_{x_2} f, \partial_{x_3} f)$ and its magnitude gives the slope of f when moving along the normal vector to $\mathcal{M}$. We remind that f is *Morse* if $k \geq 2$ and all its *critical points* $\{\mathbf{p} \in \mathcal{M} : \nabla f(\mathbf{p}) = \mathbf{0}\}$ are non-degenerate, that is, the Hessian matrix $\mathbf{H}(f) = (\partial_{x_i x_j} f)_{ij}$ of f in $\mathbf{p}$ is not singular. Given a new function $g : \mathcal{M} \to \mathbb{R}$, we compare f and g by studying the bilinear functional (see Fig. 2(a))

$$\mathcal{I}(f, g) := \langle \nabla f, \nabla g \rangle,$$

and we show its efficacy for our problem.

From the previous definition, we have that $\mathcal{I}(f, g)$ is zero at those points of $\mathcal{M}$ where ∇f is orthogonal to ∇g and at the critical points of f or g. We now characterize $\mathcal{I}$ by analyzing its critical points and estabilishing their relations with those ones of f and g. In matrix form, the gradient of $\mathcal{I}(f, g)$ may be expressed as:

$$\nabla \mathcal{I} = \mathbf{H}(f)\nabla g + \mathbf{H}(g)\nabla f. \tag{1}$$

From (1), it follows that

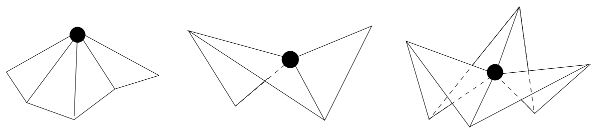

Fig. 3. Configuration of vertices around a maximum, saddle, and monkey saddle point.

- $\mathbf{p} \in \mathcal{M}$ is critical for $\mathcal{I}$ if and only if $\mathbf{H}(f)\nabla g + \mathbf{H}(g)\nabla f = \mathbf{0}$ at $\mathbf{p}$. Therefore, there might exist critical points of $\mathcal{I}$ that are not critical of f and g;
- if $\mathbf{p} \in \mathcal{M}$ is a critical point of f and g, then $\mathbf{p}$ is critical for $\mathcal{I}$;
- if $\mathbf{p} \in \mathcal{M}$ is a critical point of f and $\mathcal{I}$ but not of g, then $\mathbf{p}$ is a degenerate critical point of f.

We define as *averaged dissimilarity measure* between f and g on $\mathcal{M}$ the real number:

$$\overline{\mathcal{I}}(f,g) := \frac{1}{\text{area}(\mathcal{M})} \int_{\mathcal{M}} \langle \frac{\nabla f}{\|\nabla f\|_2}, \frac{\nabla g}{\|\nabla g\|_2} \rangle \mathrm{d}\mathbf{p}; \qquad (2)$$

finally, the *similarity measure* is defined as

$$\mathcal{I}^\star(f,g) := 1 - \frac{1}{\text{area}(\mathcal{M})} \int_{\mathcal{M}} \mid \langle \frac{\nabla f}{\|\nabla f\|_2}, \frac{\nabla g}{\|\nabla g\|_2} \rangle \mid \mathrm{d}\mathbf{p}.$$

§3. Comparison of real functions: discrete case

In this section, we present the discrete counterpart of the previously defined concepts and descriptors. We represent a compact and connected surface without boundary as a triangle mesh $\mathcal{M} := (M,T)$ where $M := \{\mathbf{p}_i,\ i = 1,\ldots,n\}$ is a set of n vertices and T is an *abstract simplicial complex* which contains the adjacency information. A function f on a triangle mesh $\mathcal{M}$ is defined by linearly interpolating the values $(f(\mathbf{p}_i))_{i=1}^n$ of f at the vertices by using barycentric coordinates. We adopt the definition of critical points defined by Banchoff [1], originally devoted to height functions defined over polyhedral surfaces and currently used by most of the computational approaches. This method uses a geometric characterization of the critical points that takes into account the position of the tangent plane with respect to the surface. A small neighborhood around a local maximum and minimum never intersects the tangent plane, while for saddles the small neighborhood is split into at least four pieces (see Fig. 3). The number of intersections r is used to associate a *discrete index* $i(\mathbf{p}, f)$ to a critical point $\mathbf{p}$ with respect to a given f. Under the assumption that the function f is *general*, i.e. $f(\mathbf{v}) \neq f(\mathbf{w})$ for all $\mathbf{v}$ and

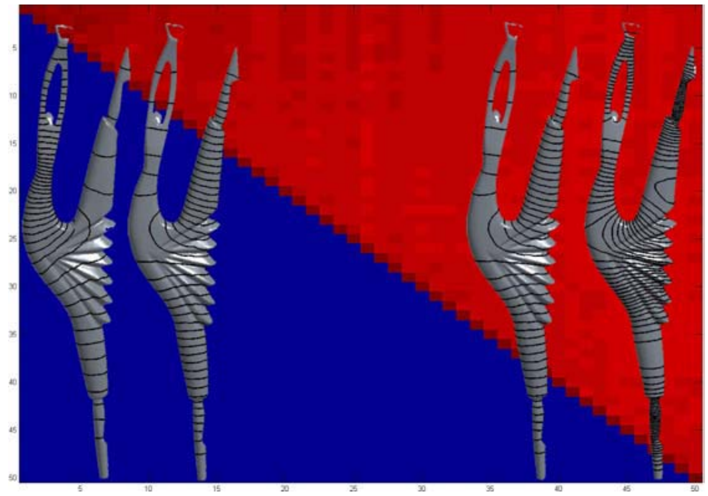

Fig. 4. Grey-scale image of the matrix $\mathbf{A}$ related to the first 50 eigenfunctions of $\mathcal{M}$; on the left (resp., right) pairs of Laplacian eigenfunctions (f_1, f_2) (resp., (f_2, f_6)) with the lowest (resp., highest) dissimilarity measure $\overline{\mathcal{I}}$.

$\mathbf{w}$ vertices of $\mathcal{M}$, critical points may occur only at those points $\mathbf{p}$ whose index $i(\mathbf{p}, f) = 1 - \frac{1}{2}r$ is not zero. In particular, the index is equal to 1 for maximum and minimum points, and can be a negative integer for saddles. For example, a monkey saddle has index -2. Finally, Banchoff proved that the relation $\sum_{\mathbf{p}\in\mathcal{M}} i(\mathbf{p}, f) = \chi(\mathcal{M})$, where χ denotes the Euclidean characteristic, holds also for polyhedral surfaces.

3.1. Dealing with two or more functions

To extend the functional (2) on a triangle mesh $\mathcal{M}$, we approximate the gradient of f at $\mathbf{p}_i$ as [6]

$$\nabla f(\mathbf{p}_i) := \sum_{j\in N(i)} [f(\mathbf{p}_j) - f(\mathbf{p}_i)]\,\mathbf{w}_j, \quad \mathbf{w}_j := \frac{1}{A_1}\mathbf{N}_1 + \frac{1}{A_2}\mathbf{N}_2 \qquad (3)$$

where $\mathbf{N}_1$, $\mathbf{N}_2$ (resp., A_1, A_2) are the normal vectors (resp., the area of the Voronoi regions) of the two triangles which share the edge (i, j), and $N(i) := \{j : (i, j) \text{ is an edge}\}$ is the 1-star of the vertex i (see Fig. 2(b) at page 3). We explicitly note that the vectors $\mathbf{w}_j$, $j \in N(i)$, do not depend on f and (3) fulfills the main properties that apply to the gradient in the continuous case, that is, linearity and nullity (i.e., f = const implies $\nabla f = \mathbf{0}$).

In several applications, we usually deal with a large number of mapping functions defined on the same surface and generated by solving Laplace equations with Dirichlet boundary conditions [8] or simulation problems (e.g., the heat equation [2]), restricting implicit functions from $\mathbb{R}^3$ to $\mathcal{M}$ (e.g., the distance from a plane or the barycentre of $\mathcal{M}$), or decomposing the spectrum of data-dependent kernels [3]. For instance, in this last case the eigenvectors of the Laplacian matrix of a triangle mesh $\mathcal{M}$ with n vertices provide up to $n-1$ non-trivial mapping functions intrinsically defined on $\mathcal{M}$. Therefore, this flexibility in the choice of f requires to generalize the previous comparison to an arbitrary number of functions on $\mathcal{M}$.

To this end, given n mapping functions $f_1, \dots, f_n$ on $\mathcal{M}$ we introduce the symmetric $n \times n$ matrix $\mathbf{A} := (a_{ij})$, where $a_{ij} := \overline{\mathcal{I}}(f_i, f_j)$, and we define as *correlation factor* of $\{f_i,\ i = 1, \dots, n\}$ the standard deviation of the set $\{a_{ij} :\ i = 1, \dots, n, j \geq i\}$, that is,

$$\sigma(f_1, \dots, f_n) := \sqrt{\frac{\sum_{i=1}^{n} \sum_{j \geq i} [a_{ij} - \overline{a}]^2}{n}}, \text{ with } \overline{a} := \frac{1}{n(n+1)} \sum_{i=1}^{n} \sum_{j \geq i} a_{ij}.$$

Since the vectors $\mathbf{w}_j$, $j = 1, \dots, n$, do not depend on the input functions, the entries of $\mathbf{A}$ can be efficiently calculated by storing the coefficients $\{\langle \mathbf{w}_i, \mathbf{w}_j \rangle :\ i = 1, \dots, n,\ j \in N(i),\ j > i\}$ and using matrix multiplications. We note that through the matrix $\mathbf{A}$ (see Fig. 4) we can identify the function f_j which mostly differs from f_i (i.e., $j = \text{argmin}_k\{|a_{ik}|\}$), as well as the pairs of functions with a similar (resp., dissimilar) behavior, i.e. (f_i, f_j) such that $a_{i,j} = \max_{p<q}\{|a_{pq}|\}$ (resp., $a_{i,j} = \min_{p<q}\{|a_{pq}|\}$).

3.2. Almost-everywhere orthogonal functions

The large amount of functions that can be defined on a given shape, as well as their high correlation, motivates the search of a minimal set of functions able to fully characterize $\mathcal{M}$. As discussed in the following, this problem naturally leads to the extraction of a function g which is "*orthogonal*" (i.e., the most dissimilar with respect to the functional $\mathcal{I}$) to a given $f : \mathcal{M} \rightarrow \mathbb{R}$. Finding such a function g is important to evaluate the degree of dissimilarity associated to a given f and to provide a locally orthogonal coordinate system on the surface, a property which is important for applications related to the computation of geodesic distances on the surface.

This problem can be reformulated as: *find* $g : \mathcal{M} \rightarrow \mathbb{R}$ *such that* ∇g *is orthogonal to* ∇f *on* $\mathcal{M}$ and we prove that it always has the trivial solution $g = \text{const}$. By combining (3) with the conditions

$$\langle \nabla f(\mathbf{p}_i), \nabla g(\mathbf{p}_i) \rangle = 0, \qquad i = 1, \dots, n, \tag{4}$$

where $\{\nabla f(\mathbf{p}_i)\}_{i=1,\dots,n}$ are constant vectors and $\{g(\mathbf{p}_i)\}_{i=1,\dots,n}$ are the unknowns, we get n linear equations

(a) (b) (c) (d)

Fig. 5. (a-b) Variation of $\overline{\mathcal{I}}$ on $\mathcal{M}$ for the pairs of mapping functions in Fig. 4; moving from white to black their similarity decreases. (c) Iso-contours of the function orthogonal to f_1; (d) critical points of f_1 and visualisation of $\overline{\mathcal{I}}$.

$$\sum_{j\in N(i)} \langle \nabla f(\mathbf{p}_i), \mathbf{w}_j \rangle g(\mathbf{p}_j) - \sum_{j\in N(i)} \langle \nabla f(\mathbf{p}_i), \mathbf{w}_j \rangle g(\mathbf{p}_i) = 0$$

with $i = 1, \ldots, n$. These relations can be written in matrix form as

$$\mathbf{Lg} = \mathbf{0} \tag{5}$$

where the entries of the $n \times n$ coefficient matrix $\mathbf{L} := (l_{ij})$ are

$$l_{ij} := \begin{cases} \langle \nabla f(\mathbf{p}_i), \mathbf{w}_j \rangle & (i, j) \text{ is an edge,} \\ \sum_{j\in N(i)} \langle \nabla f(\mathbf{p}_i), \mathbf{w}_j \rangle & i = j. \end{cases}$$

We note that the structure of $\mathbf{L}$ resembles the adjacency matrix of $\mathcal{M}$; however, in our case $\mathbf{L}$ is not symmetric and some entries might be negative. If we neglect degenerate cases where $\text{rank}(\mathbf{L}) < n - 1$, the unique solution of (5) is the vector $\mathbf{x} = \mathbf{1}$; therefore, the only functions orthogonal to f everywhere on $\mathcal{M}$ are the constant ones (see Fig. 5).

Imposing the orthogonality conditions (4) on the whole set of vertices defines only trivial solutions, which belong to the null space of $\mathbf{L}$. Our idea is to relax the orthogonality constraints by requiring $g(\mathbf{p}_i) \neq g(\mathbf{p}_j)$, for at least two distinct indices i, j. More generally, we consider the problem: *given* $I \subseteq \{1, \ldots, n\}$, *find* $g : \mathcal{M} \to \mathbb{R}$ *such that*

$$\begin{cases} \langle \nabla f(\mathbf{p}_i), \nabla g(\mathbf{p}_i) \rangle = 0 & i \in I^C, \\ g(\mathbf{p}_i) = \alpha_i & i \in I \end{cases} \tag{6}$$

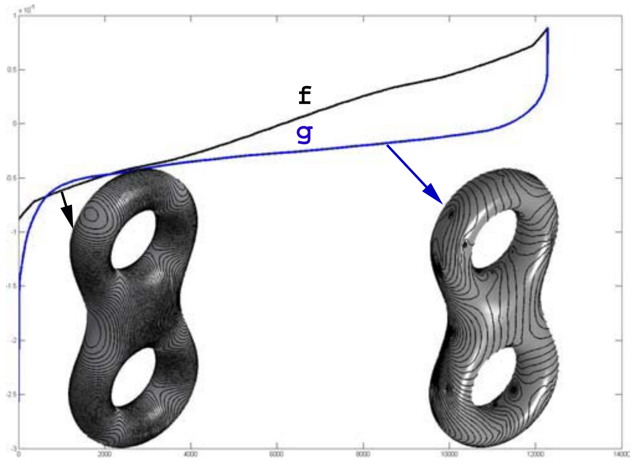

Fig. 6. The picture shows the co-domain and iso-contours of the function g orthogonal to a given f everywhere on $\mathcal{M}$ with the exception of the critical points of f.

where I^C is the complement of I. For $i \in I^C$, we rewrite (4) as

$$\sum_{j\in N(i)\cap I^C} \langle \nabla f(\mathbf{p}_i), \mathbf{w}_j\rangle g(\mathbf{p}_j) - g(\mathbf{p}_i) \sum_{j\in N(i)} \langle \nabla f(\mathbf{p}_i), \mathbf{w}_j\rangle = = - \sum_{j\in N(i)\cap I} \langle \nabla f(\mathbf{p}_i), \mathbf{w}_j\rangle g(\mathbf{p}_j). \quad (7)$$

If we assume that the cardinality of I is $n-k$, (7) is equivalent to the $(n-k)\times(n-k)$ sparse linear system $\mathbf{Lg} = \mathbf{b}$ where

$$\mathbf{L} := (l_{ij}),\ l_{ij} := \begin{cases} \langle \nabla f(\mathbf{p}_i), \mathbf{w}_j\rangle & i \in I^C,\ j \in N(I)\cap I^C \\ \sum_{j\in N(i)} \langle \nabla f(\mathbf{p}_i), \mathbf{w}_j\rangle & i = j \end{cases}$$

and

$$\mathbf{b} := (b_i)_i,\ b_i := - \sum_{j\in N(i)\cap I} \langle \nabla f(\mathbf{p}_i), \mathbf{w}_j\rangle f(\mathbf{p}_j), \quad i = 1, \ldots, n-k.$$

The sparse linear system (7) is efficiently solved by applying the conjugate gradient method [5]. Then, the hypothesis $g(\mathbf{p}_i) \neq g(\mathbf{p}_j)$, $i, j \in I$, is enough to guarantee that g is not constant; clearly, if we set $g(\mathbf{p}_i) = \alpha$, $\forall i \in I$, we achieve the solution $g = \alpha$. Once g has been calculated, the error on the orthogonality between f and g depends on the points of $\mathcal{M}$ where we did not impose the orthogonality conditions and it is equal to $\overline{\mathcal{I}}(f, g) = \frac{1}{\text{area}(\mathcal{M})} \sum_{i\in I} A_i \langle \nabla f(\mathbf{p}_i), \nabla g(\mathbf{p}_i)\rangle$, where A_i is the area of the Voronoi region of the vertex $\mathbf{p}_i$. Figures 6 and 7 show the construction of almost everywhere orthogonal functions.

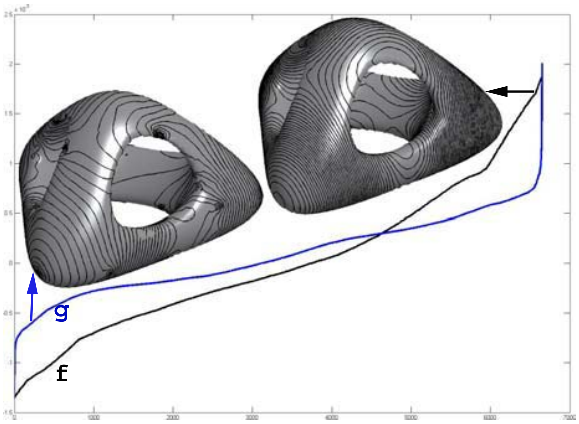

Fig. 7. Co-domain and iso-contours of the input f and its almost-everywhere orthogonal function g.

Setting up the initial values. A natural choice of I is the set of the critical points of f where the gradient vectors vanish and trivially satisfy the orthogonality conditions (6). For each $i \in I$, let $N(i)$ be its $1-$star and $\mathbf{n}_i$ be a user-defined vector (e.g., the vector orthogonal to the normal at $\mathbf{p}_i$), then $g(\mathbf{p}_i)$ is chosen as the minimum of the functional

$$G((\mathbf{p}_j)_{j\in N(i)}, \mathbf{p}_i) := \sum_{j\in N(i)} |g(\mathbf{p}_j) - g(\mathbf{p}_i)|^2$$

subject to the (discrete) linear constraints $\nabla g(\mathbf{p}_i) = \mathbf{n}_i$. If $N(i)$ has k vertices, the previous problem has $(k+1)$ unknowns and is efficiently solved by standard optimization techniques [5].

§4. Conclusions and Future Work

The paper has proposed a method able to compare and summarize into a single representation the properties of a family of functions defined on an arbitrary surface $\mathcal{M}$. The relevance of our representation is twofold: (1) the similarity measure globally estimates the independence degree of a couple of functions; (2) the bilinear functional $\mathcal{I}$ makes explicit the local differences and relationships among them. These elements make our approach able to deal with the analysis of a large number of properties derived either from the shape itself (e.g., the Laplacian eigenfunctions [3]) or from physical properties (e.g., temperature, volume, pressure [4]).

The large amount of functions defined on a given shape, as well as their high correlation, motivates the search of a minimal set of functions able to fully characterize $\mathcal{M}$. This problem naturally leads to the extraction of a function g which is everywhere orthogonal to a given $f : \mathcal{M} \rightarrow \mathbb{R}$. As we proved that the only solution to this problem is the constant function, we have provided an efficient algorithm able to construct a non-constant function g which is orthogonal to f everywhere unless its critical points. As future work, we plan to use the proposed approach to study the evolution of time-depending functions defined on the same or different surfaces.

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