Shape analysis techniques for exploring collections of 3D data

ViMac School «Visione delle Macchine»

mathematics and shapes

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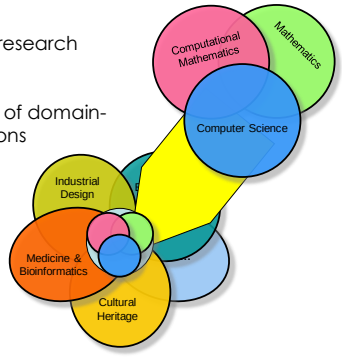
Consiglio Nazionale delle Ricerche

- The network of CNR research institutes, which are distributed all over the national territory, is multidisciplinary: it has competences in the field of health and biology, of computer science, of environment and climate, of chemistry and physics, of behavioural, economic and social sciences
- 107 institutes in 11 (now converging to **7**) departments

Istituto di Matematica Applicata e Tecnologie Informatiche, Genova
Dept. of Information and Communication Technologies

- foundational research
- development of domain-specific solutions

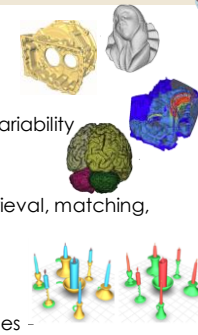


3D data
what are 3D data and 3D models

3D collections
application domains & shape variability

exploration
analysis, understanding, retrieval, matching, correspondence, … and mathematics behind

conclusions
summary & perspectives



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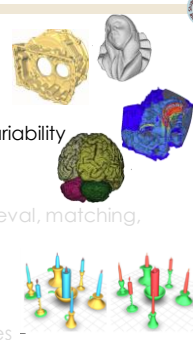


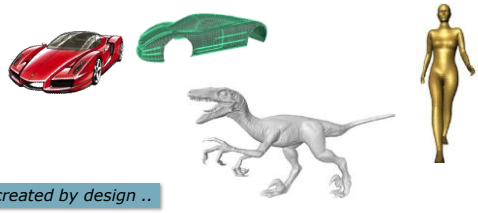
- 3D data
  - measurements of objects or phenomena characterized primarily by form and/or spatial extent in some space (two-, three- or higher-dimensional)
  - pictures, sketches, images, 3D objects, measurements of fields, videos, 4D animations, …

"Applied computer science is now playing the role which mathematics did from the seventeenth to the twentieth centuries providing an orderly, formal framework and exploratory apparatus for other sciences"

Virtual Astronomy, Information Technology and the New Scientific Methodology
George Djorgovski (2005)

3D models

- 3D data are digital representations of objects existing either in the **virtual** or physical world, and which can be processed by computer applications



.. created by design ..

3D models

- 3D data are digital representations of objects existing either in the virtual or **physical** world, and which can be processed by computer applications

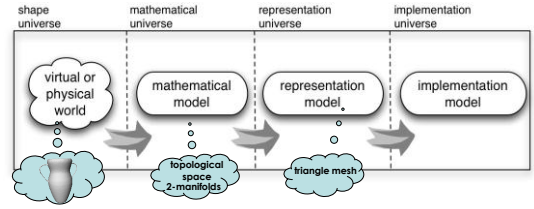
.. created by acquisition ..

geometric modelling

- developed for the CAD/CAM industry first, with focus on solid and rigid objects
- wireframe models
- 1970 – .. geometric models, such as boundary representations (B-reps) and CSG
- Terminology and definitions of the foundations of geometric modelling were introduced in the seminal work of Requicha, *[A. G. Requicha, Representations for Rigid Solids: Theory, Methods, and Systems, ACM Computing Surveys, 4(12), 1980]*

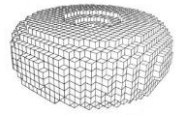
geometric modelling

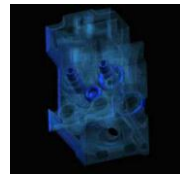
- The tradition modelling pipeline is based on an abstraction of the **geometric** nature of shapes



3D modelling

- **Cell decompositions**
  - **spatial occupancy enumeration (voxels, regular space partitioning,..)**
  - object-based (tetrahedra, octrees)
- Constructive Solid Geometry (CSG)
- Boundary models
  - triangle meshes
  - B-reps
- Sweeps
  - generalized cylinders, generic sweeps





courtesy of Tom Funkhouser

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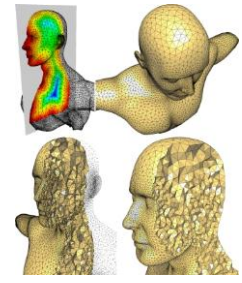

courtesy of Pierre Alliez

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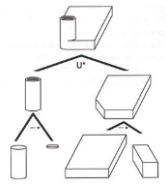

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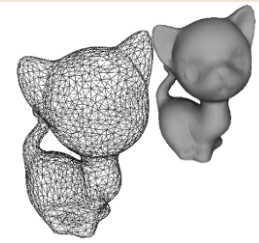

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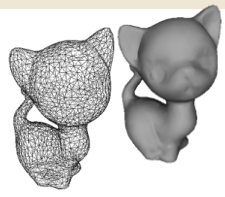

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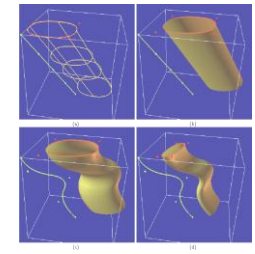
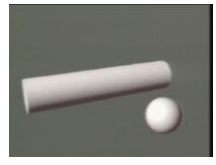


courtesy of Dimitri Terzopoulos

3D modelling

1987: from solid rigid CAD to deformable (re)-active models !!!

Dimitri Terzopoulos SIGGRAPH Award winning paper



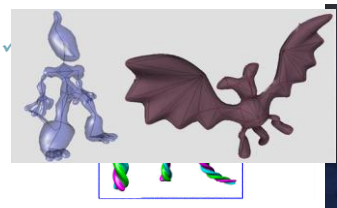
nowadays, not only rigid solids..

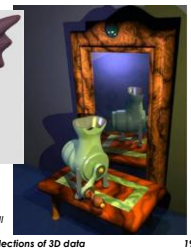
- dynamic shapes
- elastic and deformable models
- animation of natural objects and/or natural phenomena (e.g., humans, water, clouds, plants, cloths)
- from geometric models to physically-based models
  - embed semantics at some extent
- more elaborate and especially higher-level representation of the shape

nowadays, not only rigid solids..

- implicit models
  - F(x,y,z)=0
  - skeleton-based implicits

Bill Lorensen SIGGRAPH 99 Course #4 Notes





courtesy of Brian Wyvill

- "polygon soups" are very common in non-professional environments
- NURBS, B-reps, implicits, parametric, "high-quality" triangle meshes are used in professional domains for scientific and entertainment applications

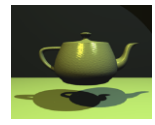
3D modelling

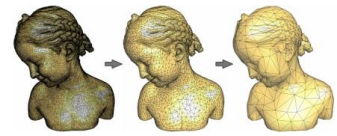
- there are many and distinct models which can be used to represent the same object digitally
- the choice of the digital representation is influenced by the application context
- each class of models has its own properties, metadata

wrt 2D data, 3D data and their models are more complex to be handled

3D shape modelling

- today, a mature domain
  - most of the fundamental problems are solved
  - plenty of solutions for modelling, analysing, compressing, streaming, processing shapes





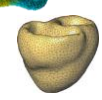
3D technology today

- the reduction of 3D hardware costs makes it possible to think now of 3D on the desktop
- 3D acquisition devices are becoming more and more commonplace
- computer networks may now rely on fast connections at low cost
- 3D displays and TVs
- 3D printers are now able to produce not only mockups but even end products

rendering, acquiring, transmitting, visualizing, and even "materializing" 3D data is now feasible also in unspecialized contexts

3D data and science

- gradual shift of paradigm in science from physical prototypes and experience to virtual prototypes and simulation
  - CAD/PLM
  - Bioinformatics
  - Medicine
  - Cultural Heritage
  - Material science
  - …



- users in these disciplines are experienced scientists or professionals but in other domains

3D data and science

- shape models are very important and visualization is "the" way to engage with the complexity of data
- not only shape data but also **knowledge**
  - what is the 3D model about?
  - in which context shape models are used?
  - what is their meaning in that context?
  - What are the characteristics of the model in its context?
- scientific disciplines are more and more converging: how is it possible to support this trend and make the "expertise" of SM more easily and transparently available to a wider community ?

3D and the internet

- from traditional multimedia to 3D content
  - 3D is widely recognized as the upcoming wave of digital media
  - The success of 3D communities and mapping applications (e.g., Second Life, HiPiHi, GoogleEarth) will cause a dramatic shift in the way people see and navigate the Internet

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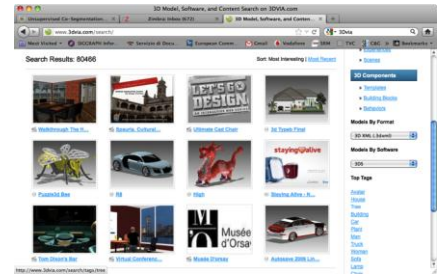
3D and the internet

- Dassault 3DVia: user-generated 3D content and collections



shape modelling

- likely to become the basis of tomorrow's networked and collaborative platforms
  - for scientific and industrial domains that deal with 3D (or n-D) scientific data
  - for networked entertainment and virtual/gaming applications
- as an enabling technology, it will have to face and meet the needs of many users & contexts
  - professionals and non-3D-experts

exploring 3D collections

- .. but how to navigate this large amount of information and data?
- data and knowledge expressed by 3D content is simply useless if it cannot be easily manipulated, accessed, retrieved and re-used

… a grand challenge!

exploring 3D collections

- .. but how to navigate this large amount of information and data?
- Exploration by
  - Analysis and understanding
  - Classifying
  - Searching and retrieving
  - Matching
  - Finding correspondences
  - …

shape analysis and segmentation

- what do we want to capture with the segmentation ?

Many different kinds of shapes…

Many different *parts of interest* in a shape…

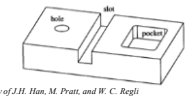
…. that vary according to the context and shape type

shape analysis and segmentation

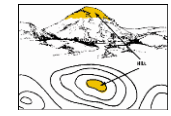
- automatic process, different for different representation models
- identification of pre-defined features classes
  - slots, pockets, ridges ..
- identification of parts, or surface patches
  - geometric criteria, e.g. curvature
  - approximation of primitives, e.g. geons
  - statistical approaches, e.g. learning of features
- target understanding and supports several applications
  - skeletonization, parametrization, compression
  - descriptors and similarity

shape analysis and segmentation

- form feature recognition in CAD/CAM
  - Lyc Kyprianou. *Shape classification in Computer-Aided Design*, Ph.D. Dissertation, Cambridge university, 1980
  - A feature is a region of a part with some interesting geometric or topological patterns
  - Pratt M.J. and Wilson P.R., *Requirements for support of form features in a solid modeling system*, CAM-I, R-85-ASPP-01, 1985
- morphological features in spatial data handling
  - Pfaltz, J. L. 1976 *Surface networks*. Geographical Analysis 8,.
  - Nackman, L. 1984. *Two-dimensional Critical Point Configuration Graphs*. PAMI,6, 4.



Courtesy of J.H. Han, M. Pratt, and W. C. Regli

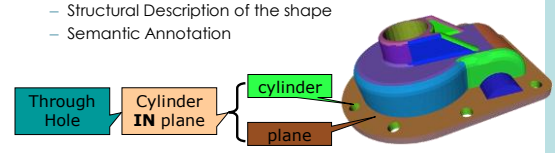


shape analysis and segmentation

- the majority of methods developed for 3D shape analysis rely on a "light" reference to the **semantics** of the segments or parts
- "geons"-like [Biederman,87]
  - a shape is perceived/understood as a compound of **primitives with simpler shape**
- minima rule [Hoffmann&Richards, 83]
  - lines of **minimal curvature** on the boundary of the shape identify the *natural* boundaries of the relevant visual parts of a shape

shape analysis and segmentation

- theories of human perception show that humans "understand" shapes by recognizing interesting sub-parts and their structure [Marr 1982, Biederman 1987]
- for specific contexts, it is possible to describe what these "features" are and how they are structured.
  - Geometric Description of the features
  - Structural Description of the shape
  - Semantic Annotation



mesh segmentation

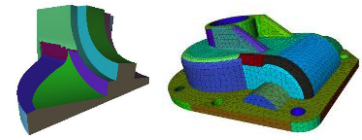
- given a mesh M = {V,E,F}, a partition on F (could be also on V or E) into *k* disjoint subsets induces a segmentation of M in a set of sub-meshes

- the boundaries of the segments are usually edges of the original mesh
- segments do not overlap

courtesy and © of Ariel Shamir

segmentation based on Fitting Primitives (HFP)

- Automatic generation of a binary tree of triangle clusters, each fitted to one of the available primitives: planes, spheres and cylinders
  - At the first step, each triangle is considered to be a cluster
  - Then, adjacent triangles are clustered according to cluster-to-primitive fitting



Feature Points and Core Extraction (FPCE)

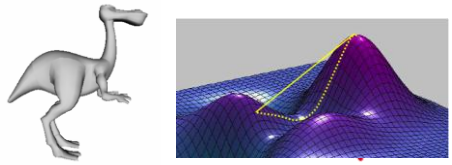
- hierarchically and **pose-invariant** segmentation into"perceptually relevant" parts of an articulated shape

Feature Points and Core Extraction (FPCE)

- hierarchically
  - first, simplify and segment the whole shape
  - then, iteratively, simplify and segment each sub-mesh

Feature Points and Core Extraction (FPCE)

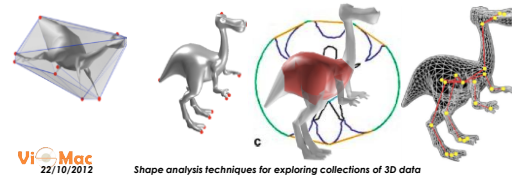
- pose-invariance
  - transform the shape into a pose-invariant representation and use this to segment the mesh (and sub-meshes afterwards)



multi-dimensional scaling

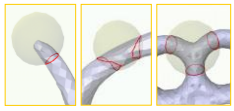
Feature Points and Core Extraction (FPCE)

- feature-point and core extraction
  - pick points of the trasformed shape that touch its convex hull => features
  - mirror-transform the transformed shape
  - use points that lie on the convex hull => boundaries between shape components (along minima of curvature..)

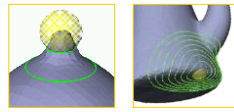


multi-scale **curvature** and **morphology** (Tailor)

- the label of vertex *v* at scale *i* is given by the configuration of the intersection between the mesh and a sphere having radius R_i and centred in *v*.



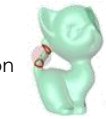
different number of connected components in the intersection line

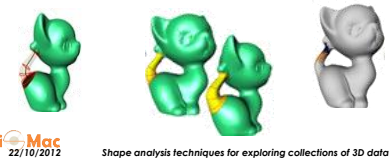


evolution of the intersection lines through scale variation

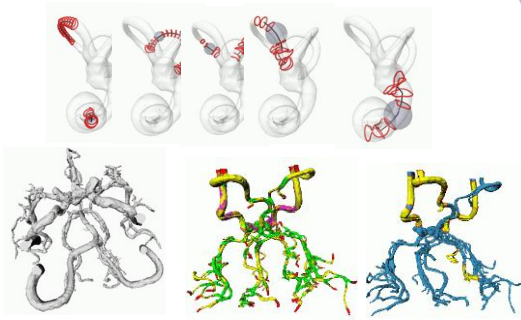
tubular parts (Plumber)

- Plumber, the tube finder
  - selection of the scale R
  - classification of vertices and identification of seed limb region
  - medial loop identification
  - tubular feature extraction
  - skeletonization of tubular features



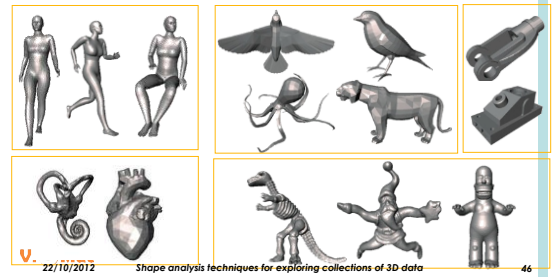


tubes are important!



comparison

- selection of 14 models of different types
  - virtual humans (3), animals (4), CAD (2), medical (2), miscellanea (3)



comparison

- evaluation criteria
  - type of segmentation (*semantics* vs *geometry* oriented methods)
  - correctness (*qualitative assessment*)
  - boundaries (*smooth* vs *jagged*)
  - hierarchical vs multi-scale
  - sensitivity to pose (articulated objects)
  - sensitivity to noise and/or tessellation
  - asymptotic complexity
  - control parameters

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correctness (med.)

FPCE TAILOR PLUMBER HFP

correctness (CAD)

FPCE TAILOR PLUMBER HFP

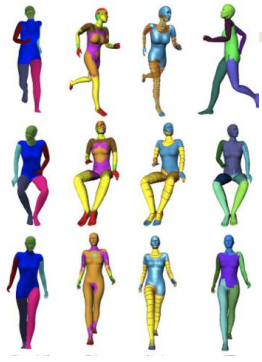
correctness (animals)

FPCE TAILOR PLUMBER HFP

correctness (misc)

FPCE TAILOR PLUMBER HFP

correctness (VH)



FPCE TAILOR PLUMBER HFP

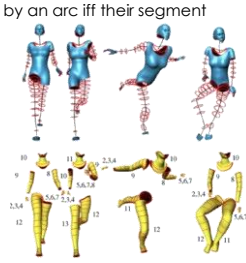
from geometry to semantics

- ✓ the decomposition provided by Plumber is coded in an abstract graph where each component is a node with metric attributes
  - cylindrical (resp, conical) parts
    - scale of the tube
    - set of cross sections and their average radius (resp, basis radius)
    - set of centreline points
    - orientation of each centreline segment
    - axis length and component volume
  - body parts
    - number of boundaries
    - volume
    - refined segmentation

from geometry to semantics

- the decomposition provided by Plumber is coded in an abstract graph where each component is a node with metric attributes
  - nodes in the graph are joined by an arc iff their segment boundaries are adjacent

using anthropometric studies, the ratio and range of the various attributes allows for the automatic classification of the components as body parts



references

- Hierarchical segmentation using fitting primitives: primitives with pre-defined geometry,"geons"-oriented
  - M. Attene, B. Falcidieno, and M. Spagnuolo. Hierarchical Mesh Segmentation based on Fitting Primitives. The Visual Computer, 22, 2006
- Segment articulated objects in its main components: primitives identified by their boundaries, no fixed geometry
  - S. Katz, G. Leifman, and A. Tal. Mesh Segmentation using Feature Points and Core Extraction. The Visual Computer, 21:8-10, 2005
- From geometric to semantic segmentations: use context-information to segment & annotate the shape
  - M. Mortara, G. Patané, M. Spagnuolo, B. Falcidieno, and J. Rossignac. Blowing Bubbles for the Multi-scale Analysis and Decomposition of Triangle Meshes. Algorithmica, Special Issues on Shape Algorithms, 38(2), 2004 M.
  - M. Mortara, G. Patanè, and M. Spagnuolo. From geometric to semantic human body models, Computers & Graphics, 30(2), pp: 185-196, 2006

remarks

- each algorithm is suited to specific classes of object/features
  - shape segmentation algorithms are generally not really contextualized even if they make implicitly assumptions on the class of shapes they are targeting
- evaluation should be based on *quantitative* but also on *qualitative* assessments
- shape segmentation ground-truth should be defined
  - more shape classes, probably more specialized (e.g., four legged animals vs fishes) or classified differently (e.g., articulated vs non-articulated)
- shape segmentation should try to make better use of the context

shape segmentation benchmark

- quantitative analysis of how people decompose objects into parts and for comparison of automatic mesh segmentation algorithms
  - 80 people manually segmented surface meshes into functional parts, yielding an average of 11 human-generated segmentations for each of 380 meshes across 19 object categories
  - this data set provides a sampled distribution over "**how humans decompose each mesh into functional parts**", which we treat as a probabilistic ground truth
  - Number of Segments, Segment Area, Cut Perimeter, Min Curvature, Max Curvature, Mean Curvature, Gaussian Curvature, Curvature Derivative, Dihedral Angle, Convexity, Compactness

Xiaobai Chen, Aleksey Golovinskiy, and Thomas Funkhouser,
A Benchmark for 3D Mesh Segmentation
ACM Transactions on Graphics (Proc. SIGGRAPH), 28(3), 2009.

Shape segmentation benchmark

Xiaobai Chen, Aleksey Golovinskiy, and Thomas Funkhouser,
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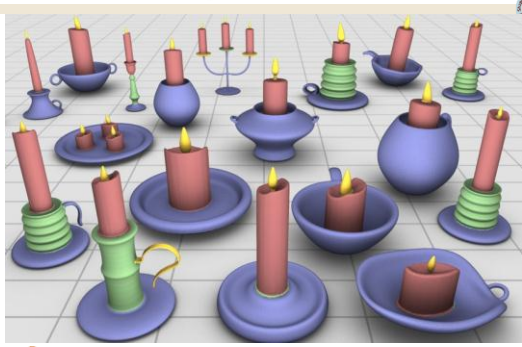
co-segmentation of 3D collections

- is it possible to extract more information about a set shapes by analyzing them simultaneously, rather than analyzing just one shape or a pair of shapes at a time?

co-segmentation of 3D collections



co-segmentation of 3D collections

at the end...

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...thank you very much for your attention ...

.. and enjoy the second part!

these notes will be available at: http://www.ge.imati.cnr.it/training

Shape analysis techniques for exploring collections of 3D data

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exploring by evaluating similarities

outline

- ✓ 3D data
  - 3D models and representations
  - complexity, variability, contexts
- ✓ Exploration of 3D collections
  - Exploring by analysis and segmentation
  - Exploring by similarity for retrieval, classification, and matching
- ✓ Mathematics
- ✓ Conclusions

exploring 3D collections

- ✓ 3D data are characterized not only by **shape** but also **knowledge**
- ✓ search&retrieval is necessary for supporting
  - analysis and classification *(understanding)*
  - content creation and re-use
- ✓ **similarity** is the notion behind these tasks

objects and similarity

objects and similarity

geometric congruence

structural equivalence

functional equivalence

semantic equivalence

objects and similarities

geometric congruence

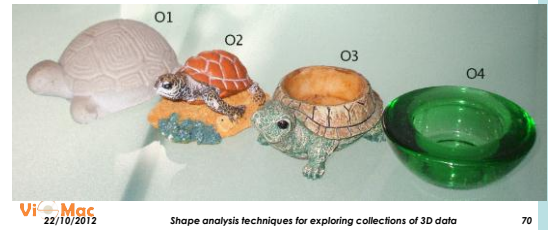
structural equivalence

functional equivalence

semantic equivalence

experiment n. 1

- ✓ look at these objects and freely describe them
  - are they similar ?
  - with respect to what?
  - in what sense?



mathematics: shape description and similarity

- ✓ similar shapes with **respect to what**?
  - shape descriptions, to code the aspects of shapes to be taken into account and manage the complexity of the problem
- ✓ **similarity** in what sense ?
  - transformations among the shapes that we consider irrelevant to the assessment of the similarity
    - invariants or properties

mathematics: shape description and similarity

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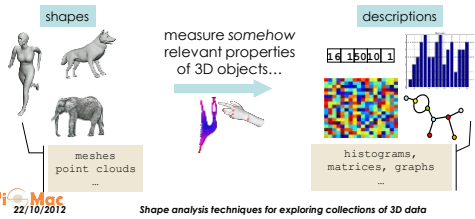
- ✓ **similarity** in what sense ?
  - transformations among the shapes that we consider irrelevant to the assessment of the similarity
    - invariants or properties

shape and description

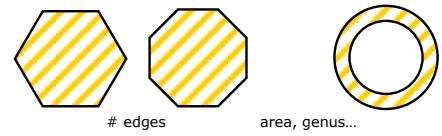
- shape descriptions reduce the complexity of the representation; their choice depends on
  - **type** of shapes and their variability/complexity
  - **invariants** or properties



shape descriptions

- in general, a description could be just a set of numbers…

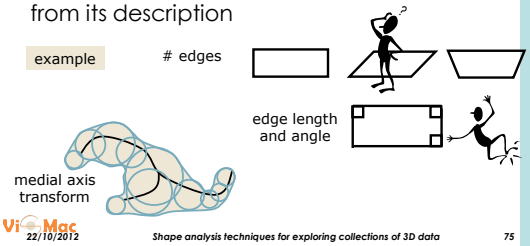
example



shape descriptions

- different shapes should have different descriptions
  - different enough to discriminate among shapes
- a shape may not be entirely reconstructed from its description

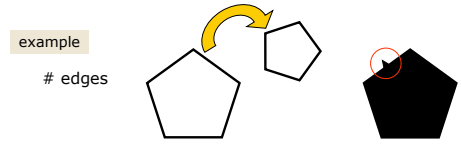
example



shape descriptions

- invariance
- uniqueness
- stability to noise
- sensitivity to global/local features

example



invariance

- invariance = the descriptor does not change for a given object under a class of transformations
- a property P is invariant to a transformation T applied to an object O iff

$$P(T(O)) = P(O)$$

example

mathematics: shape description and similarity

- similar shapes with **respect to what**?
  - shape descriptions, to code the aspects of shapes to be taken into account and manage the complexity of the problem
- **similarity** in what sense ?
  - transformations among the shapes that we consider irrelevant to the assessment of the similarity
    - invariants or properties

intuition vs mathematics

- congruence
  - two objects are congruent if one can be transformed into the other by rigid movements (translation, rotation, reflection – **not scaling**)

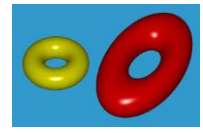
intuition vs mathematics

- similarity
  - two geometrical objects are called *similar* if one can be obtained by the other by uniform stretching . Formally, a *similarity* of a Euclidean space S is a function $f: S \rightarrow S$ that multiplies all distances by the same positive scalar r, so that:
    $d(f(x), f(y)) = rd(x, y), \forall x, y \in S$

intuition vs mathematics

- affinity
  - it preserves collinearity, i.e. maps parallel lines into parallel lines and preserve ratios of distances along parallel lines
  - it is equivalent to a linear transformation followed by a translation

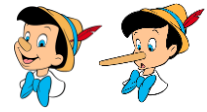
transformations and similarities



affine transformation

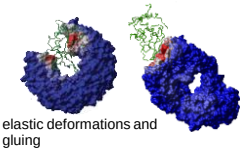
image from http://cse.taylor.edu/~btoll/s99/424/res/mtu/Notes/geometry/geo-tran.htm

isometric transformation



"locally-affine" transformation

Images from http://www.disneyclips.com/, © Disney copyright, all rights reserved



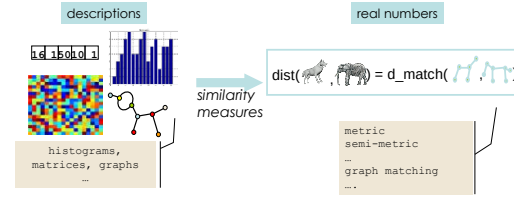
elastic deformations and gluing

experiment n. 2

- use in turn O1, O2, O3 and O4 as queries
- rank the remaining three objects with respect to the query, according to your notion of similarity

shape descriptions and similarity

- *similarity* in what sense ?
  - defining appropriate *similarity measures* between shape descriptions



distance measures

- let **S** be a set of shape descriptors, the distance measure **d** between two shapes descriptors is defined as:

$$d: S \times S \rightarrow \mathbb{R}$$

- properties:
  - $d(x,x)=0$ (self identity)
  - $d(x,y) \geq 0, x \neq y$ (positivity)
  - $d(x,y)=d(y,x)$ (symmetry)
  - $d(x,z) \leq d(x,y)+d(y,z)$ (triangular inequality)
  - $d(x,y) \leq \max\{d(x,z), d(z,y)\}$ (strong t. i.)

distance measures

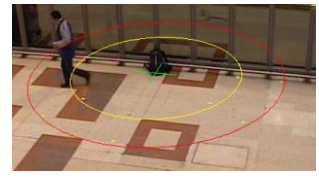
- the measure properties are grouped as in the following:
  - semi-metric: self-identity, positivity, symmetry
  - pseudo-metric: self-identity, symmetry, triangular inequality
  - metric: a pseudo-metric that satisfies the positivity
  - ultra-metric: a metric satisfying strong triangular inequality
- the perceptual space can be approximated by the metric properties ? *[Tversky, Psychological Review N 84, 1977], [Santini et al. PAMI 1999, Vol 21, N 9,]*
  - symmetry and triangular inequality should not hold for partial matching
- metrics can be used for indexing purposes

shape matching issues

- shape descriptors
  - properties the shape descriptors should satisfy (e.g. expressiveness invariance, robustness)
- distance measures
  - properties (e.g. intuitive notion of similarity, indexing)
- shape matching
  - properties the shape matching methodologies should satisfy (e.g. effectiveness, efficiency)
    - which information carried by the shape has to be used?
    - which methodology has to be used to exploit this information?

3D matching

- several application contexts deal with problems that can be solved by comparing 3D objects:
  - **security**
    - face recognition (is this person allowed to walk in this area?)
    - detect dangerous situations (is this unattended object a baggage or a trolley?)

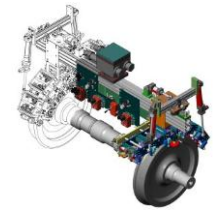


3D matching

- several application contexts deal with problems that can be solved by comparing 3D objects:
  - **gaming & simulation**
    - build virtual worlds and characters (is it available a specific object already designed by others?)
    - simulate dangerous situations (is it available an avatar that can be animated?)

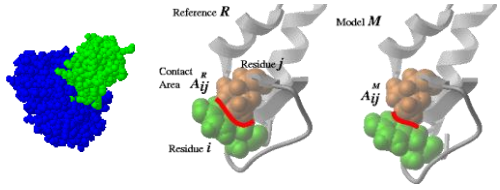
3D matching

- several application contexts deal with problems that can be solved by comparing 3D objects:
  - **industrial manufacturing**
    - design a new mechanical part starting from an existing one



3D matching

- several application contexts deal with problems that can be solved by comparing 3D objects:
  - **medicine & biology**
    - drug design (is it available a molecular model belonging to a specific category?)



3D matching

- application contexts:
  - security
  - gaming & Simulation
  - industrial manufacturing
  - medicine & biology

- matching tasks
  - recognition
  - retrieval
  - classification
  - docking

shape recognition

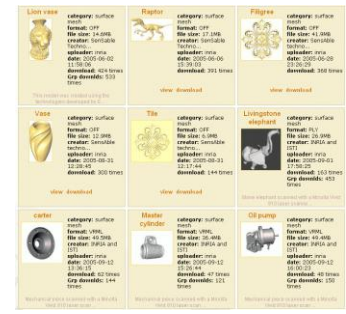
- given a query shape ***S***, does ***S*** belong to the repository ?

- aims the search at a *specific* shape
  - **precise match**
- examples: security, drug design, copyright protection

shape retrieval

- given a query shape ***S*** does the repository contain an object **similar** to ***S*** ?

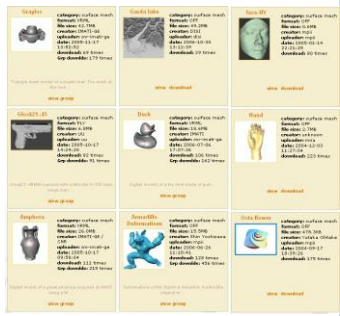
- aims the search at a *category* level
  - **similarity assessment**
- examples: security, design, gaming & simulation



shape classification

- given a classified repository and a query shape ***S***, find the class ***S*** belongs to

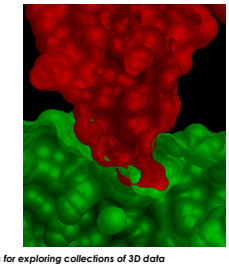
- aims the match at a *category* level
  - **similarity evaluation**
- examples: security, bioinformatics, medical applications



molecular docking

- molecular docking can be thought of as a problem of "*lock-and-key*",
  - given a set of keys the problem is to find the correct "key" that opens up a "lock"
  - the protein can be thought of as the "lock" and the ligand can be thought of as a "key"

- takes into account **complementarity of shapes**
- the Protein and the ligand **change their shape** while interacting



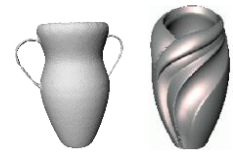
- examples:
  - drug design

3D matching

- matching tasks:
  - recognition
  - retrieval
  - classification
  - docking
- matching approaches :
  - global matching
  - partial matching

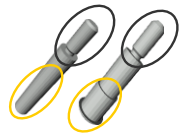
global matching

- overall shape comparison
  - the object as a whole
- real number representing the similarity estimation between the two objects
  - distance measure
  - properties



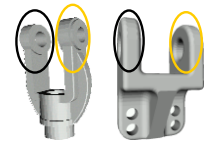
part correspondence

- real number as similarity estimation
  - distance measure
  - properties
- mapping among similar parts
  - similar overall shape
  - identification of the relevant parts of the objects
  - explicit representation of the correspondence among object parts



partial matching

- real number as similarity estimation
  - distance measure
  - properties
- similar parts between objects having different overall shape
  - identification of the relevant parts of the objects
  - the two objects share similar parts
  - explicit representation of the correspondence among object parts
  - highlight of shape differences



Shape analysis techniques for exploring collections of 3D data

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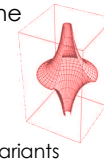
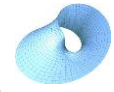
mathematics and shape similarity

things are not that easy…

- the simple examples we have shown are clearly not enough to deal with the *complexity* at a hand…
- we need tools to reason about
  - connectivity, interior, exterior and boundary
  - measuring shape properties and invariants
  - well-posedness
  - robustness and stability
  - distance and proximity
  - etc…





content

- ✓ mathematical concepts
  - topological spaces
  - continuous and smooth functions
  - homeo- and diffeomorphisms
  - manifolds
  - metric spaces
  - curvature
  - Riemannian surfaces
  - geodesic distances
  - Laplace-Beltrami operator
  - Gromov-Hausdorff distance
- ✓ concepts in action
  - surface correspondence
  - intrinsic shape description
  - Reeb graphs

content

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  - Reeb graphs

why topological spaces?

- ✓ to represent the set of observations made by the observer (e.g., neighbor, boundary, interior, projection, contour);
- ✓ to reason about stability and robustness

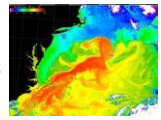
topological spaces

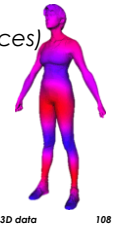
- ✓ a *topological space* is a set X together with a collection T of subsets of X, called *open sets*, satisfying the following axioms:
  1. $X, \emptyset \in T$
  2. any union of open sets is open
  3. any finite intersection of open sets is open

- ✓ the collection T is called a *topology* on X

why functions?

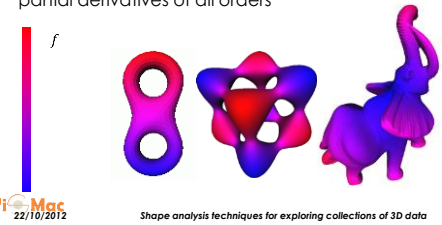
- ✓ to characterize shapes
- ✓ to measure shape properties
- ✓ to model what the observer is looking at
- ✓ to reason about stability
- ✓ to define relationships *(e.g., distances)*





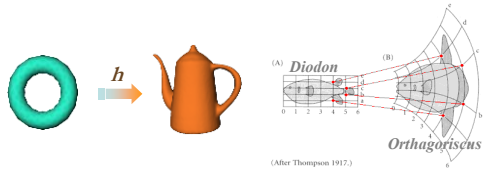
continuous and smooth functions

- ✓ let X, Y topological spaces, $f : X \to Y$ is *continuos* if for every open set $V \subseteq Y$ the inverse image $f^{-1}(V)$ is an open subset of X
- ✓ let X be an arbitrary subset of $\mathbb{R}^n$; $f : X \to \mathbb{R}^m$ is called *smooth* if $\forall x \in X$ there is an open set $U \subseteq \mathbb{R}^n$ and a function $F: U \to \mathbb{R}^m$ such that $F = f_{|X}$ on $X \cap U$ and F has continuous partial derivatives of all orders



homeo- & diffeo- morphisms

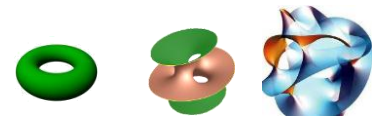
- a *homeomorphism* between two topological spaces and is a continuous bijection with continuous inverse



- given and , if the smooth function is bijective and is also smooth, the function is a *diffeomorphism*

why manifolds?

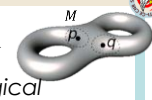
- to formalize shape properties
- to *ease* the analysis of the shape
  - measuring properties *walking on the shape*
  - look at the shape *locally* as if we were in our traditional euclidean space
  - to exploit *additional geometric* structures which can be associated to the shape

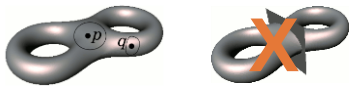


Images courtesy of D. Gu and Jbourjai on Wikimedia Commons

manifold

- manifold without boundary a *topological Hausdorff space* M is called a *k-dimensional topological manifold* if each point $q \in M$ admits a neighborhood $U_i \subseteq M$ homeomorphic to the open disk $D^k = \{x \in \mathbb{R}^k \,|\, |x| < 1\}$ and $M = \bigcup_{i \in \mathbb{N}} U_i$



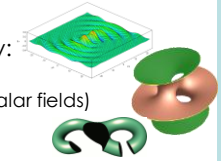


- k is called the dimension of the manifold

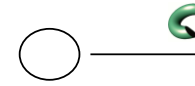
examples

- 3-manifolds with boundary:
  - a solid sphere, a solid torus, a solid knot
- 2-manifolds:
  - a sphere, a torus

- 2-manifold with boundary:
  - a sphere with 3 holes, single-valued functions (scalar fields)

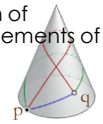


- 1 manifold:
  - a circle, a line



metric space

- a *metric space* is a set where a notion of distance (called a metric) between elements of the set is defined



- formally,
  - a **metric space** is an ordered pair (X, d) where X is a set and d is a metric on X (also called distance function), i.e., a function

$$d: X \times X \to \mathbb{R}$$

such that $\forall x, y, z \in X$:

- $d(x, y) \geq 0$; *(non-negative)*
- $d(x, y) = 0$ iff $x = y$; *(identity)*
- $d(x, y) = d(y, x)$; *(symmetry)*
- $d(x, z) \leq d(x, y) + d(y, z)$ *(triangle inequality)*

what properties and invariants?

- how far are $\boldsymbol{p}, \boldsymbol{q}$ on X and $\boldsymbol{p}', \boldsymbol{q}'$ on Y?

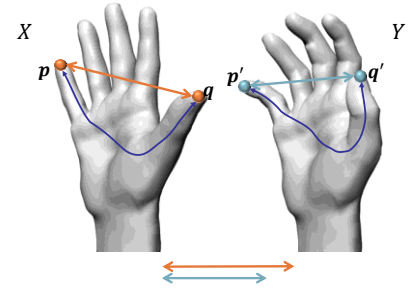
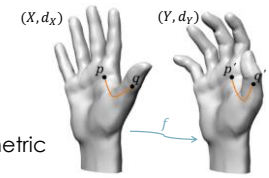


image partially from: Bronstein A. et al. PNAS 2006;103:1168-1172

isometries

- an *isometry* is a bijective map between metric spaces that preserves distances:



- looking for the right metric space…
  - the Euclidean distance
  - geodesic distances, diffusion distances, …

image partially from: Bronstein A. et al. PNAS 2006;103:1168-1172

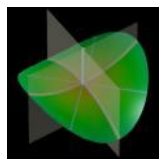
invariance and isometries

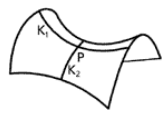
- a property invariant under isometries is called an *intrinsic* property

- examples:
  - the Gaussian curvature K
  - the first fundamental form
  - the geodesic distance
  - the Laplacian operator

principal curvatures

- the principal curvatures measure the maximum and minimum bending of a surface at each point along lines defined by the intersection of the surface with planes containing the normal





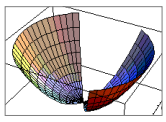
Gaussian and mean curvature

- given k_1 and k_2 the principal curvatures at a point surface
  - *Gaussian curvature* $K = k_1 k_2$
  - *mean curvature* $H = (k_1 + k_2)/2$

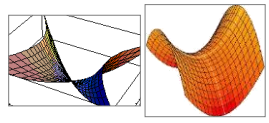
- according to the behavior of the sign of K, the points of a surface may be classified as
  - elliptic
  - hyperbolic
  - parabolic or planar

examples

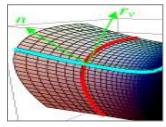
-



K>0



K<0

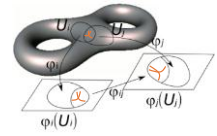


$K = 0, H \neq 0$

conformal structure and Riemann surfaces

- a conformal structure is a structure assigned to a topological manifold such that angles can be defined

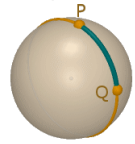
- a topological surface with a conformal structure is called a *Riemann surface*



- a 2-manifold (real) can be turned into a Riemannian surface iff it is orientable and metrizable
  - the Möbius strip, the Klein bottle, the projective plan do not admit a conformal structure

geodesic distance

- minimal geodesics: shortest path between two points on the surface
- geodesic distance between P and Q: length of the shortest path between P and Q
- geodesic distances satisfy all the requirements for a metric
- a Riemannian surface carries the structure of a metric space whose distance function is the geodesic distance



Laplace-Beltrami operator

- continuous case

$$\Delta f := \mathrm{div}(\mathrm{grad} f)$$

- Laplacian eigenvalue problem

$$\Delta f = -\lambda f$$

- orthonormal eigensystem

$$\mathcal{B} := \{(\lambda_i, \psi_i)\}_i \qquad \Delta\psi_i = \lambda_i \psi_i$$

$$\lambda_0 \leq \lambda_1 \leq \ldots, \lambda_i \leq \lambda_{i+1} \ldots \leq +\infty$$

discrete Laplacian operator

$$\Delta f(\mathbf{p}_i) := \frac{1}{d_i} \sum_{j \in N(i)} w_{ij} \left[f(\mathbf{p}_i) - f(\mathbf{p}_j) \right]$$

- $N(i)$ index set of 1-ring of vertex $\mathbf{p}_i$
- $f(\mathbf{p}_i)$ function value at vertex $\mathbf{p}_i$
- d_i mass associated with vertex $\mathbf{p}_i$
- w_{ij} edge weights

discrete geometric Laplacian

- Desbrun et al. (1999)

$$w_{ij} := \frac{\cot(\alpha_{ij}) + \cot(\beta_{ij})}{2} \qquad d_i := a(i)/3$$

  - the cotangent weights take into account the angles opposite to edges, the masses take into account the area around vertices
- Meyer et al. (2002)

$$d_i := a_V(i)$$

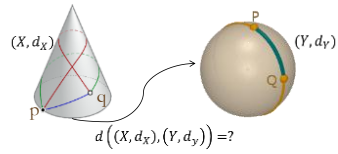
  - cotangent weights, masses considering the Voronoi area
- Belkin et al. (2003, 2008)
  - weights constructed using heat kernels
- Reuter et at. (2005, 2006)
  - weak formulation of the eigenvalue problem

$$\langle \Delta f, \varphi_i \rangle_{\mathcal{L}^2(\mathcal{M})} = -\lambda \langle f, \varphi_i \rangle_{\mathcal{L}^2(\mathcal{M})}$$

  with φ_i cubic form functions

metrics between spaces

- the Gromov-Hausdorff distance poses the comparison of two spaces as the direct **comparison of pairwise distances** on the spaces
- equivalently, it measures the distortion of embedding one metric space into another



Gromov-Hausdorff distance

- let $(X, d_X), (Y, d_Y)$ be two metric spaces and $C \subset X \times Y$ a correspondence, the *distortion* of C is:

$$dis(C) = \sup_{(x,y),(x',y') \in C} |d_X(x, x') - d_Y(y, y')|$$

- the Gromov-Hausdorff distance is

$$d_{GH}(X, Y) = \frac{1}{2} \inf_C dis(C)$$

- variations: *Lp Gromov-Hausdorff* distances and *Gromov-Wasserstein* distances

properties

- ✓ the Gromov-Hausdorff distance is parametric with respect to the choice of metrics on the spaces X and Y
- ✓ common choices
  - Euclidean distance (estrinsic geometry)
  - geodesic distance (intrinsic geometry) or, alternatively, diffusion distance

$$d^2{}_{X,t}(x,y) = \sum_{i=0}^{\infty} e^{-2\lambda_i t}(\psi_i(x) - \psi_i(y))^2$$

where (λ_i, ψ_i) is the eigensystem of the Laplacian operator and t is time

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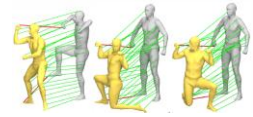
- ✓ A. M. Bronstein, M. M. Bronstein, R. Kimmel, M. Mahmoudi, G. Sapiro. *A Gromov-Hausdorff Framework with Diffusion Geometry for Topologically-Robust Non-rigid Shape Matching*. *Int. J. Comput. Vision* 89, 2-3, 266-286, 2010

content

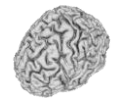
- ✓ mathematical concepts
  - topological spaces
  - continuous and smooth functions
  - homeo- and diffeomorphisms
  - manifolds
  - metric spaces
  - curvature
  - Riemannian surfaces
  - geodesic distances
  - Laplace-Beltrami operator
  - Gromov-Hausdorff distance
- ✓ concepts in action
  - surface correspondence
  - intrinsic shape description
  - Reeb graphs

surface correspondence

- ✓ problem: finding correspondences between a discrete set of points on two surface meshes
  - *extrinsic* vs *intrinsic* correspondence



- ✓ why: **attribute transfer, surface tracking, shape analysis (brain imaging)**, …

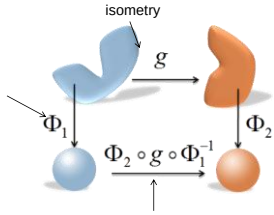


Courtesy of Michael Bronstein

intrinsic correspondence [Lipman and Funkhouser 2009]

- ✓ looking for an **intrinsic** correspondence means finding corresponding points such that the mapping between them is close to an **isometry**
- ✓ idea:

any genus zero surface can be mapped conformally to the unit sphere

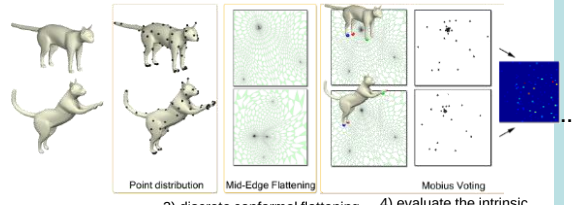


1-1 and onto conformal map of a sphere to itself (Mobius map): uniquely defined by three corresponding points

intrinsic correspondence [Lipman and Funkhouser 2009]

1) sampling points: local maxima of Gauss curvature & (geodesically) farthest point algorithm

3) compute the Möbius transformation that aligns a triplet in the common domain



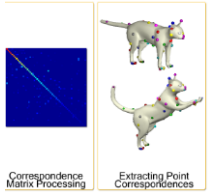
2) discrete conformal flattening to the extended complex plane

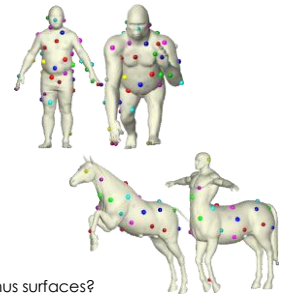
4) evaluate the intrinsic deformation error and build a fuzzy correspondence matrix

intrinsic correspondence [Lipman and Funkhouser 2009]

5) produce a discrete set of correspondences

...

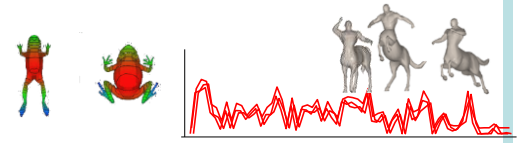




- pay attention to…
  - what about higher genus surfaces?
  - drawbacks of the discrete (linear) flattening technique

deformable shape characterization

- problem: describing intrinsic shape properties, and deriving signatures



courtesy of Michael Bronstein

- why: shape registration, global and partial matching, …

intrinsic shape description [Sun et al. 2009]

- *heat equation*, governing the distribution of heat from a source on a surface X; initial conditions: heat distribution at time $t = 0$:

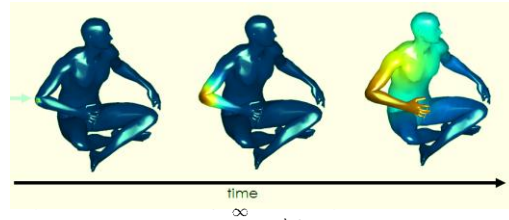
$$\left(\Delta_X - \frac{\partial}{\partial t}\right) u = 0$$

- the heat kernel is a fundamental solution of the heat equation with point heat source at x (and heat value at y after time t)

$$K_t(x,y) = \sum_{i=0}^{\infty} e^{-\lambda_i t}\phi_i(x)\phi_i(y)$$

$$\approx \sum_{i=0}^{N} e^{-\lambda_i t}\phi_i(x)\phi_i(y)$$

intrinsic shape description [Sun et al. 2009]



$$K_t(x,y) = \sum_{i=0}^{\infty} e^{-\lambda_i t}\phi_i(x)\phi_i(y)$$

$$\approx \sum_{i=0}^{N} e^{-\lambda_i t}\phi_i(x)\phi_i(y)$$

courtesy of Michael Bronstein

intrinsic shape description [Sun et al. 2009]

- heat kernel signature as a point description over the temporal domain:

$$HKS(x): \mathbb{R}^+ \to \mathbb{R}, \qquad HKS(x,t) = k_t(x,x)$$

- multiscale, informative, intrinsic, localized sensitivity to topological noise

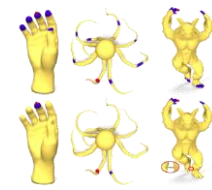
- distance between signatures at scale $[t_1, t_2]$:

$$d_{[t_1,t_2]}(x,x') = \left(\int_{t_1}^{t_2} \left(\frac{|k_t(x,x) - k_t(x',x')|}{\int_M k_t(x,x)dx}\right)^2 d\log t\right)^{1/2}$$

courtesy of Michael Bronstein

intrinsic shape description [Sun et al. 2009]

feature point detection



localizing repeated structures at different scales

- pay attention to…
  - multiscale… but how to handle time scale?
  - discretization and robustness issues

references

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- Lipman, Y., and Funkhouser, T. 2009. Mobius voting for surface correspondence. *ACM Transactions on Graphics* 28(3) *(SIGGRAPH)*
- Sun, J., Ovsjanikov, M., and Guibas, L. 2009. A concise and provably informative multi-scale signature based on heat diffusion. In *Proc. of the Symp. on Geometry Processing*, Eurographics Association, Aire-la-Ville, Switzerland, Switzerland, SGP '09,1383–1392
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isometries, but not only…

- topological spaces and functions allow us to model shapes and measures/properties about shapes as pairs (X, f)
- this settings allows us to talk about stability and robustness

- isometries allows us to study a *important/relevant* set of problems related to shape and similarity…
- *.. are there alternatives to isometries that allow us to solve other sets of problems?*

about transformations (again)

- several transformations $f: X \to X'$ that can be applied to a space X

congruence ⊂ — homothety ⊂ similarity ⊂ affinity ⊂ / isometry ⊂ quasi-isometry ⊂ — Möbius transformation ⊂ — homeo-morphism

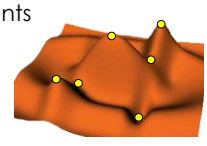
are these transformations enough to describe a shape and its relation to other shapes? what else? can we define other invariants?

an alternative…

- algebraic topology associates algebraic invariants to each space so that **two spaces are homeomorphic if they have the same invariants**
- approach: to **decompose** a topological space **into simple pieces** that are easier to study (e.g. to decompose a polyhedron into faces, edges, vertices or a surface into triangles)

…more transformations

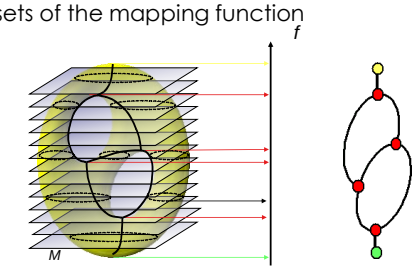
- what if we want to reason about shapes under more invariants?
- **critical points** of functions may give good characterizations of shape properties which reflect different invariants



…stay tuned…
see the M. Ferri's and C. Landi's talks

Reeb graph

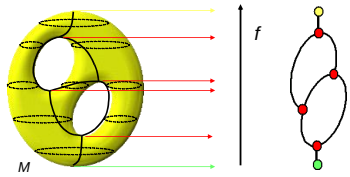
- Reeb graphs store the evolution of the level sets of the mapping function



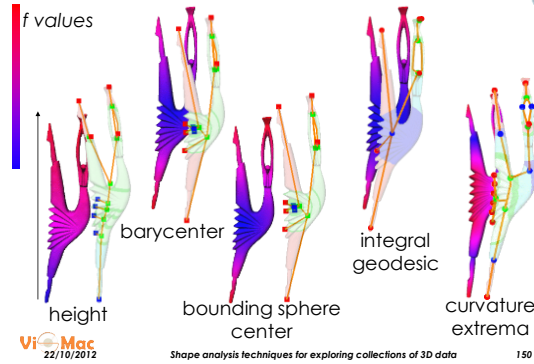
Reeb graph definition

- given defined on the surface , the Reeb graph of wrt is the quotient space defined by :

& are in the same connected component of

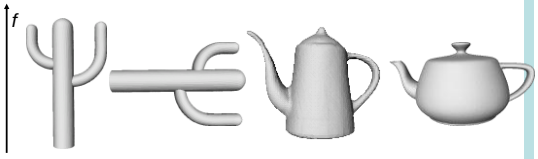


overview of RGs when the function *f* varies

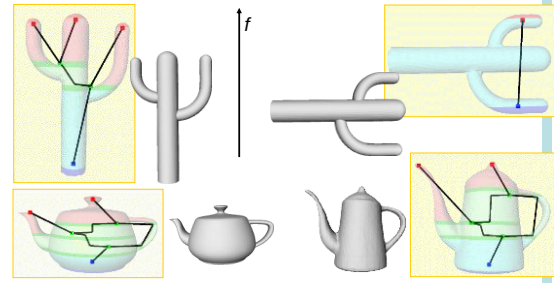


quiz

- Draw the Reeb graph with respect to the height function f of the following shapes

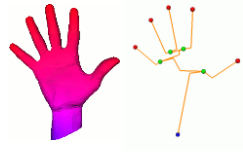


quiz - solutions



RG properties

- it provides a 1D structure of the shape
- it describes the shape of an object under the lens of the function f
- nodes and arcs depend on the number of critical points of f
- it may be computed in $O(n \log n)$ operations



Reeb graph based representations

- Reeb graph variations
  - contour trees (simply-connected domains)
  - component trees (gray-level images)
  - centerline skeletons (geodesic distance from a point)
- for shape matching
  - Multiresolution Reeb graph (MRG), Hilaga et al. 2001
  - augmented Multiresolution Reeb graph (aMRG), Tung&Schmitt 2005
  - Extended Reeb graph (ERG), Biasotti et al. 2000

references

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Shape analysis techniques for exploring collections of 3D data

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wrap up

to sum up: theory says …

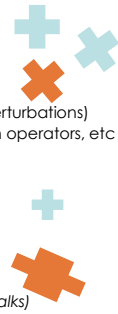
- ✓ we have seen
  - how to represent a shape (topological spaces, manifolds, metric spaces, simplicial complexes…)
  - how to analyze, describe, compare shapes (geodesics, curvature, diffusion geometry, etc…)
  - examples of different applications
- ✓ we have investigated
  - several tools for several problems (best choices for space, invariants, properties…)
  - theoretical assumptions vs real world assumptions
- ✓ what's next?
  - mathematics in the forthcoming landscape

to sum up: theory says …

- ✓ 3D shapes maybe very complex and services we can imagine for sharing, searching, and even design new shapes are many, so…
- ✓ … how to get out of this maze?
- ✓ basics
  - the choice of one or another shape description should be guided by
    - type of shapes and their variability/complexity
    - invariants or properties to be preserved / captured
  - topological spaces and functions are promising to model and reason about similarity in a mathematically well-formulated manner

pay attention to…

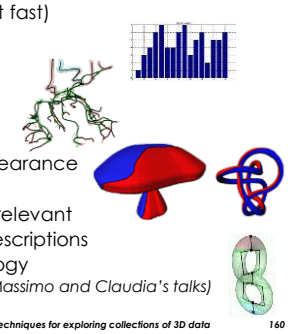
- ✓ … the right space
  - rigid transformations (rotations, translations)
    - Euclidean distances
  - isometries
    - Riemannian metric
    - curvature (but unstable to local noise/perturbations)
    - geodesics, diffusion geometry, Laplacian operators, etc
  - local invariance to shape parameterizations
    - conformal geometry
  - similarities (i.e. scale operations)
    - normalized Euclidean distances
  - affinity (and homeomorphisms)
    - Morse theory
    - persistent topology
    - size theory *(see Massimo and Claudia's talks)*



pay attention to…

- ✓ … a suitable shape description
  - coarse coding (but fast)
    - histograms
    - matrices
  - articulated shapes
    - medial axes
    - Reeb graphs
  - overall global appearance
    - silhouettes
  - if shape loops are relevant
    - graph-based descriptions
    - persistent topology

(see Massimo and Claudia's talks)



to sum up: benchmarking

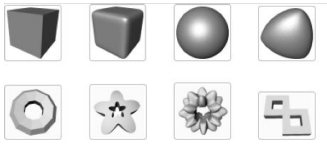
- ✓ importance of benchmarking and evaluation!
  - not only publishing papers but also to demonstrate real innovation and applicability potential
- ✓ shape segmentation benchmark
- ✓ SHape Retrieval Context (SHREC) http://www.aimatshape.net/event/SHREC
  - an annual event to to evaluate the effectiveness of 3D shape analysis algorithms
  - a multi-track event spanning
    - different models: from watertight objects to triangle soups, from abstract shapes to medical data
    - different tasks: from 3D retrieval to correspondence finding and segmentation

example: SHREC'12 - SAS

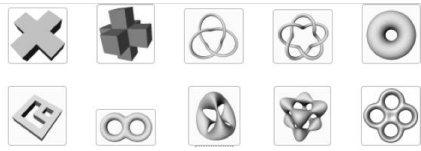
- ✓ to evaluate the stability of the algorithms under variations of abstract shapes characterized by
  - smooth/sharp features,
  - partial/global symmetries
  - different genus.
- ✓ to create a new benchmark where to measure the capability of the retrieval methods to be invariant to
  - noise addition and resampling
  - affine transformations

dataset

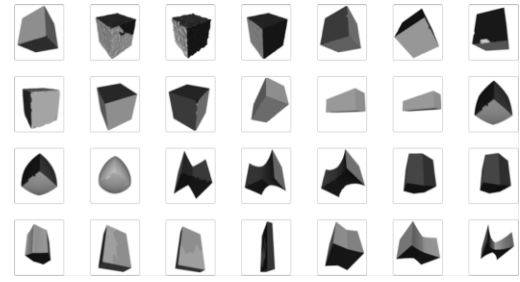
9 transformations, 3 degrees each, for a total of 27 perturbations



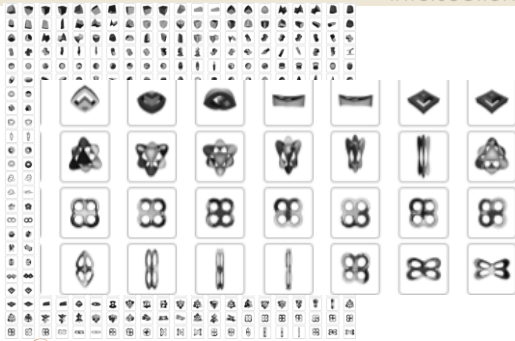
1. addition of Gaussian noise
2. different regular sampling
3. uneven sampling
4. & .5 two stretching
6. & .7 two non-uniform dilations
8. & .9 two non-uniform erosions



dataset



504 watertight models without self-intersections



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- ✓ ***INTEROMICS:*** *Development of an integrated platform for the application of "omic" sciences to biomarker definition and theranostic, predictive and diagnostic profiles*, Flagship project, 2011-2013
- ✓ the Shape Modelling group at IMATI
  - B. Falcidieno, M. Attene, C. Catalano, M. Mortara, G. Patanè, F. Robbiano
- ✓ D. Giorgi, M. Ferri, C. Landi, P. Frosini

at the end…

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…thank you very much for your attention!

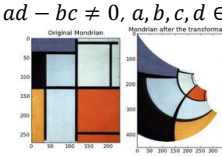
these notes will be available at: http://www.ge.imati.cnr.it/training

Möbius transformation

- ✓ is a rational function $f: \mathbb{C} \cup \{\infty\} \rightarrow \mathbb{C} \cup \{\infty\}$:

$$f(z) = \frac{az+b}{cz+d}$$

such that $ad - bc \neq 0$, $a, b, c, d \in \mathbb{C}$



  - is a bijective conformal map of the Riemann sphere to itself
  - also known as *homographic*, *linear fractional*, *bilinear*, or *fractional linear transformation*