The Hitchhiker's Guide to the Galaxy of
Mathematical Tools for Shape Analysis

SIGGRAPH 2012 *Course Notes*

Mathematical Guide (Part 2)

content

- ✓ mathematical concepts
  - basics on algebraic topology
  - simplicial complexes
  - homology
  - surface genus
  - critical points
  - Morse theory
- ✓ concepts in action
  - persistent topology
  - Reeb graphs

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isometries, but not only…

- ✓ topological spaces and functions allow us to model shapes and measures/properties about shapes as pairs (X, f)
- ✓ this settings allows us to talk about stability and robustness
- ✓ isometries allows us to study a *important/relevant* set of problems related to shape and similarity..
- ✓ *.. are there alternatives to isometries that allow us to solve other sets of problems?*

about transformations (again)

- ✓ several transformations $f: X \to X'$ that can be applied to a space X

congruence ⊂ | homothety ⊂ similarity ⊂ affinity ⊂ / isometry ⊂ quasi-isometry ⊂ | Möbius transformation ⊂ | homeo-morphism

are these transformations enough to describe a shape and its relation to other shapes? what else? can we define other invariants?

why algebraic topology?

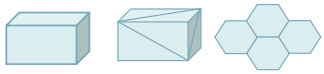
- ✓ algebraic topology associates algebraic invariants to each space so that **two spaces are homeomorphic if they have the same invariants**
- ✓ approach: to **decompose** a topological space **into simple pieces** that are easier to study (e.g. to decompose a polyhedron into faces, edges, vertices or a surface into triangles)

basics on algebraic topology

- a combinatorial structure is generated by the decomposition of the topological space
- basic elements of the decomposition are cells or simplices that are characterized by
  - combinatorial aspect: relations among the cells of the complex
  - geometric aspect: related to their embedding in the Euclidean space

examples

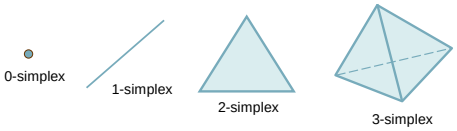
- cells complexes
  - a geographic map (which is made of points, lines and regions)
  - a decomposition of a polyhedron into faces



- simplicial complexes
  - e.g., triangle meshes

simplex

- a k-*simplex* in a Euclidean space $\mathbb{R}^n$, with $n \geq k$, is the convex hull of $k+1$ affinely independent points. a subset of these points defines a simplex of dimension $< k$ called *face*.
- in $\mathbb{R}^3$



simplicial complex

- a (finite) simplicial complex K is a (finite) collection of simplices so that
  - if $\sigma \in K$ and τ is a face of σ, then $\tau \in K$
  - if $\sigma_1, \sigma_2 \in K$, then $\sigma_1 \cap \sigma_2$ is a face of both (that is, two simplices can only meet along a common face)
- the *dimension* of K is the highest among the dimensions of its simplices
  - triangle meshes are 2-complexes
  - tetrahedral meshes are 3-complexes

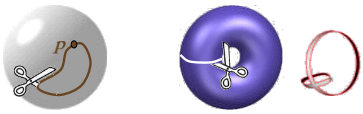
chains and boundaries

- a k –chain is a linear combination of all k simplices in K, $\sigma = \sum_i \lambda_i \sigma_i$, $\lambda_i \in \mathbb{Z}$
  - a curve on a mesh is a 1-chain
  - a surface patch is a 2-chain

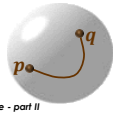
- the n-dimensional *chain space* is a linear space made by all the n-chains
- the boundary operator extracts the boundary of the chain

loops

- a *loop* (1-*cycle*) is a closed curve whose initial and final points coincide in a fixed point p known as the *basepoint*
  - a closed 1-chain is a loop

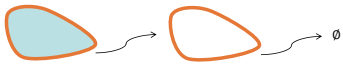


  - a non closed chain has boundary vertices

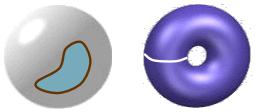


exact and closed chains

- the boundary of a boundary is empty

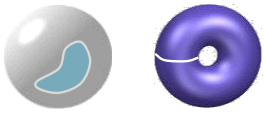


- a k-chain γ is exact, if there exists a $(k+1)$-chain σ, such that γ is the boundary of σ
  - exact chains are closed
  - reverse is not true



homology

- intuition: the difference between the closed chains and the exacts chains indicates the topology of a space
  - any closed 1-chain on genus zero surface is exact
  - on tory, some closed 1-chains are not exact



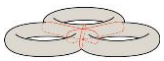
- (simplicial) homology formalizes this idea

simplicial homology

- the *q-th simplicial homology group* of K, $H_q(K)$, is the quotient group of cycles modulo boundaries
  - an element of $H_q(K)$ is an equivalence class, called *homology class*, of homologous q-cycles, that is, cycles whose difference is a boundary
- the rank of $H_q(K)$ is called the *q-th Betti number* of K, and it is a measurement of the number of different holes in K
  - for 3D data the three Betti numbers β_0, β_1, and β_2 count the number of connected components, tunnels, and voids, respectively
- the *homology* $H_*(K)$ is a topological invariant of *K*

genus

- the *genus* g of a surface S without boundary is:
  - half the first Betti number of S
  - the cardinality of a minimal set of mutually non-isotopic loops which can be cut along the surface without disconnecting it

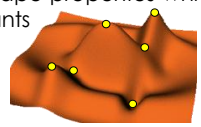


- any orientable surface without boundary is a connected sum of g tori, where $g \geq 0$ is its genus
- the genus of a surface with boundary is the genus of the surface S' obtained by gluing a disc onto each boundary component

- the genus of a surface is a topological invariant

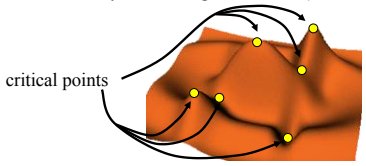
algebraic topology and differential geometry

- algebraic topology and invariants to homeomorphisms
- what if we want to reason about shapes under more invariants?
- **critical points** of functions may give good characterizations of shape properties which reflect different invariants



functions and critical points

- given a smooth function $f: M \to \mathbb{R}$ on a smooth manifold M, a point x is called
  - *regular* if the differential df_x is surjective
  - *critical* if df_x is the zero map
- a critical point is called
  - *non-degenerate* if the Hessian matrix *H* of the second partial derivatives of f is non singular at that point



critical points

- formally:
  - a point P is critical for f if:

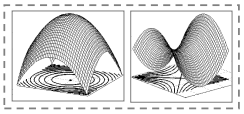
$$\frac{\partial f}{\partial x_1}(p) = 0, \frac{\partial f}{\partial x_2}(p) = 0, \dots, \frac{\partial f}{\partial x_k}(p) = 0$$

  - It is Morse if:

$$\left| H_f(p) \right| = \left| \frac{\partial^2 f}{\partial x_i \partial x_j}(p) \right| \neq 0$$

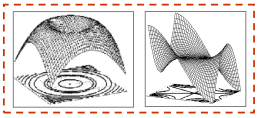
- If x is a non-degenerate critical point of f, the number λ of negative eigenvalues of H is called the index of x

critical points on a surface



non-degenerate c. p.

degenerate c. p.



Morse theory

- a function f is called *Morse* if all of its critical points are non-degenerate
- Morse theory studies the relationship between a function's critical points and the topology of its domain
- it indicates when the topological type changes and what kind of changes take place
- it provides a surface decomposition into a limited set of primitive topological cells, defined by the surface critical points and their corresponding index

does any Morse function exist?

- Morse functions are everywhere dense in the space of all smooth functions on the manifold
- any Morse function has only a finite number of critical points on a compact manifold
- the set S of all simple Morse functions is everywhere dense in the set of all Morse functions
- examples of Morse functions on a smooth manifold: *height function, distance functions, geodesic distance*, etc.

Morse theory & critical points

- Let C_i = #{critical points of index i} and β_i the i-th Betti number of M; then
  - Weak Morse inequalities
    - $\beta_i \leq C_i$
    - $\sum_i (-1)^i C_i = \sum_i (-1)^i \beta_i = \chi(M)$
  - Strong Morse inequalities
    - $\forall i \geq 0,\ \beta_i - \beta_{i-1} + \cdots + \beta_0 \leq C_i - C_{i-1} + \cdots + C_0$

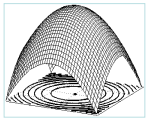
$$\#maxima - \#saddles + \#minima = \chi(M) = 2 - 2g$$

Morse theory & critical point configuration

- (Morse Lemma) In a neighbourhood of each non-degenerate critical point P, the function f can be expressed as:

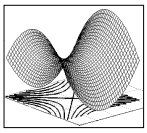
where λ is the index of the critical point

2-manifolds



$f = -x^2 - y^2$

maximum
λ=2



$f = -x^2 + y^2$

saddle
λ=1

3-manifolds

$f = +x^2 + y^2 + z^2$

minimum
λ=0

$f = -x^2 - y^2 - z^2$

maximum
λ=3

$f = -x^2 + y^2 + z^2$

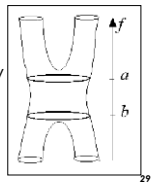
saddle
λ=1

$f = -x^2 - y^2 + z^2$

saddle
λ=2

Morse theory & critical points

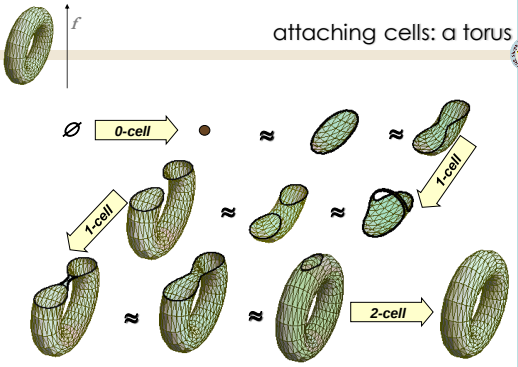
- let $f: M \to \mathbb{R}$ be a real valued function and let $[a,b] \subset \mathbb{R}$ be an interval non containing critical values of f; then the level sets $f^{-1}(a)$ and $f^{-1}(b)$ are diffeomorphic
- denote $M^x = \{p \in M | f(p) \leq x\}$ and P a critical point such that $f(p) = c$; then:
  $\forall \varepsilon > 0$ such that $f^{-1}[c-\varepsilon, c+\varepsilon]$ contains no other critical points of f, the set $M^{c+\varepsilon}$ has the homotopy type of $M^{c-\varepsilon}$ with a λ-cell attached



Morse theory & shape decomposition

- Theorem (CW complex decomposition)
  - let S be a smooth compact manifold embedded in an Euclidean space. Let $f: S \to \mathbb{R}$ be a smooth, real valued, Morse function on S. Then S is homeomorphic (i.e. topologically equivalent) to a cell complex of dimension i for each critical point of index i

attaching cells: a torus



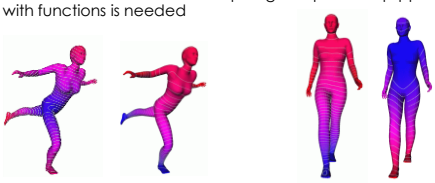
pictures from http://www.cs.rug.nl/~gert/topology.html

Morse theory can be used for..

- to combine the topology of X with the quantitative measurement provided by f
  - f is the lens to look the properties of (X, f)
  - different choices of f provide different invariants
- to find CW structures and handle decompositions on manifolds and to obtain substantial information about their homology
- to construct a general framework for shape characterization which if parameterized wrt the pair (X, f)

about distances (again)

- to assess **how far** two shapes, in this settings (X, f) and (Y, g), are
  - a notion of metric between topological spaces equipped with functions is needed



- **natural pseudo-distance**: shapes are similar if there exists a homeomorphism between the spaces that preserves the properties conveyed by the functions

natural pseudo-distance

- given two shapes (X, f) and (Y, g), the natural size pseudodistance is defined as the infimum of

$$\Theta(\gamma) = \max_{P \in X} \left\| \phi(P) - \psi(\gamma(P)) \right\|_{\infty}$$

as γ varies in the set of all the homeomorphisms from X onto Y

- it is small iff that induces a small change on the function f, that is, if there exists an homeomorphism mapping one space into the other while preserving the properties conveyed by the real function
- how to compute it? stay tuned...

references

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content

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  - loops on a surface
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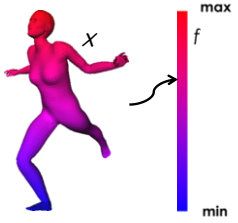
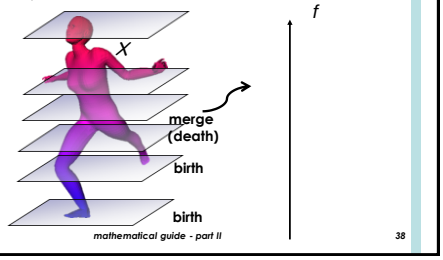
Morse Theory and computational topology



a *shape* is a pair (X, f), that is, a topological space X endowed with a real function $f: X \to \mathbb{R}$ describing its properties

persistent topology

- topological events *[e.g. birth, merge of connected components]* occur while sweeping X through f, i.e., when analysing X as the evolution of its sub-level sets $X_t = \{P \in X: f(P) < t\}$, $t \in \mathbb{R}$



persistent topology

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basic idea:
encode the lifespan of topological events; lifespan is proportional to the importance of the features they represent: long events stand for significant features, short events stand for either details or noise

an historical perspective

- different topological features:
  - size functions: [Frosini 1991], 0-th degree homology (connected components)
  - persistent homology: [Edelsbrunner et al. 2000], higher degree homology (cycles)
- from 1-dimensional to multidimensional properties:
  - 1-dimensional setting: $f: X \to \mathbb{R}$ [Frosini 1991, Edelsbrunner et al. 2000]
  - multidimensional setting: $\vec{f}: X \to \mathbb{R}^k$ [Biasotti et al. 2008]

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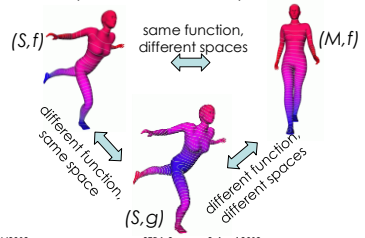
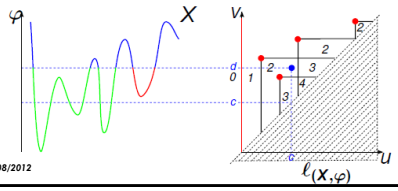
size theory and size functions [Frosini 1991]

- in size theory two shapes are similar if their natural pseudo-distance is zero
- aim: to provide ***computable*** bounds to the homeomorphisms between spaces and functions



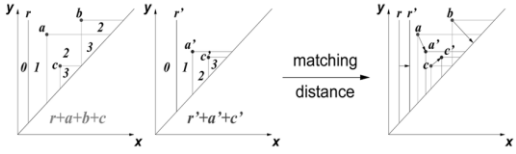
size functions

- the size function $\ell_{(X,\varphi)}: \{(u,v) \in \mathbb{R}^2: u < v\} \to N$ takes each $(u,v)$to the number of connected components of X_v that contain at least one point of X_v
  - that is, $\ell_{(X,\varphi)}(u,v)$ = number of connected components (0-homology classes) born before u and still alive at v



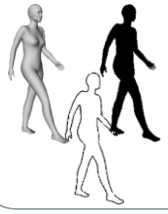
size functions and distances

- size functions can be represented as countable collections of points and lines of the plane with multiplicities (named cornerpoints and cornerlines)
- the matching *(chamfer)* distance between size functions is *stable* and is a *lower bound* of the natural pseudo-distance



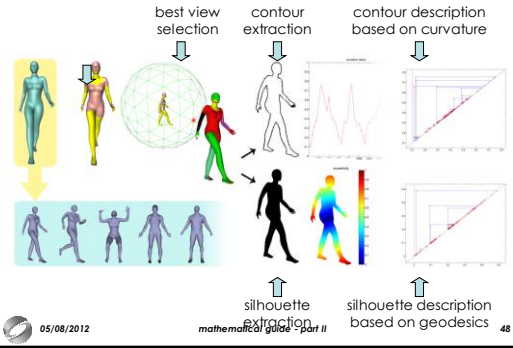
modularity

- parametric wrt the choice of the space *X*



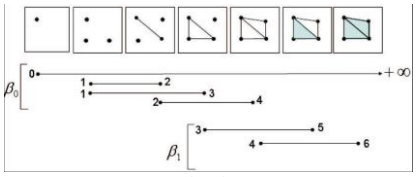
- parametric wrt the choice of the function *f*

size functions and view-based 3D retrieval [Mortara et al. 2010]



persistent homology (barcodes)

- given the pair , consider the collection of nested lower level sets of
- measure the scale at which a topological feature (e.g., a connected component, a tunnel, a void) is created, and when it is annihilated along this filtration using homology groups
- encode this information as parametrized Betti numbers



barcodes: matching [Carlsson et al. 2005]

- examples on mathematical surfaces
- classification results on a set of 80 hand-drawn copies of letters

an historical perspective

- different topological features:
  - Size Functions: [Frosini 1991], 0-th degree homology (connected components)
  - Persistent Homology: [Edelsbrunner et al. 2000], higher degree homology (cycles)
- from 1-dimensional to multidimensional properties:
  - 1-dimensional setting: $\varphi : X \to \mathbb{R}$ [Frosini 1991, Edelsbrunner et al. 2000]
  - multidimensional setting: $\vec{\varphi}: X \to \mathbb{R}^k$ [Biasotti et al. 2008]

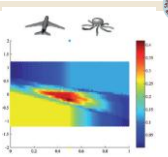
multidimensional setting

- why a vector-valued measuring function?
- many properties are intrinsically multidimensional *(coordinates, colour...)*; alternatively, we may want to blend the information of different one-dimensional properties

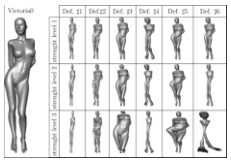
X (X, φ_1) (X, φ_2) (X, φ_3) ...

multidimensional setting: (still) open issues

- need of efficient algoritms to evaluate the size function [Biasotti et al. 2011]
- shape comparison of textured shapes

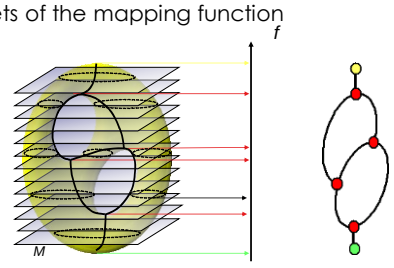


Dmatch						
	0.00	1.59	7.77	7.77	23.54	23.99
	1.59	0.00	7.77	7.77	24.71	25.15
	7.77	7.77	0.00	3.42	22.63	23.08
	7.77	7.77	3.42	0.00	20.07	20.51
	23.54	24.71	22.63	20.07	0.00	1.25
	23.99	25.15	23.08	20.51	1.25	0.00



Reeb graph

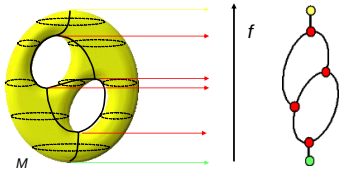
- Reeb graphs store the evolution of the level sets of the mapping function



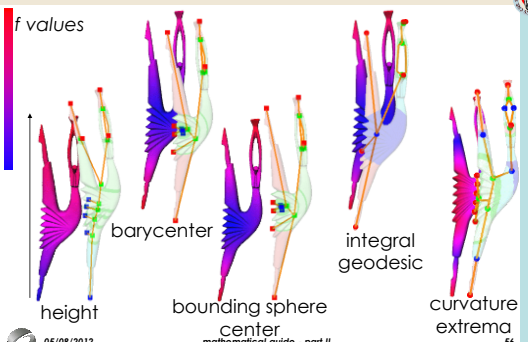
Reeb graph definition

- given f defined on the surface M, the Reeb graph of M wrt f is the quotient space defined by "$\sim_R$":

$P, Q \in M$, $P \sim_R Q \Leftrightarrow f(P) = f(Q)$ and P, Q are in the same connected component of $f^{-1}(f(P))$

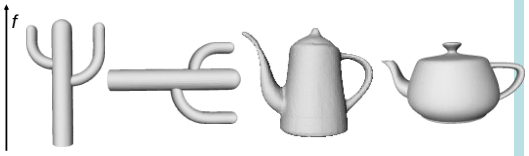


overview of RGs when the function f varies

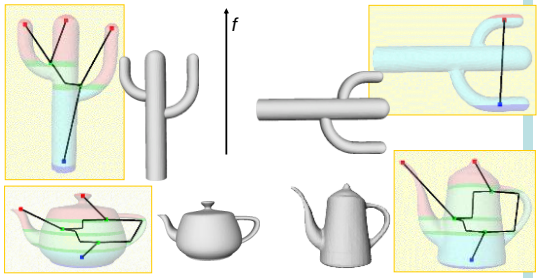


quiz

- draw the Reeb graph with respect to the height function f of the following shapes

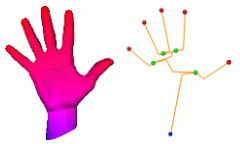


quiz - solutions



RG properties

- it provides a 1D structure of the shape
- it describes the shape of an object under the lens of the function f
- nodes and arcs depend on the number of critical points of f
- it may be computed in $O(n \log n)$ operations



Reeb graph based representations

- Reeb graph variations
  - contour trees (simply-connected domains)
  - component trees (gray-level images)
  - centerline skeletons (geodesic distance from a point)
- for shape matching
  - Multiresolution Reeb graph (MRG), Hilaga et al. 2001
  - augmented Multiresolution Reeb graph (aMRG), Tung&Schmitt 2005
  - Extended Reeb graph (ERG), Biasotti et al. 2000

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