The Hitchhiker's Guide to the Galaxy of Mathematical Tools for Shape Analysis

SIGGRAPH 2012 *Course Notes*

Mathematical Guide (part 1)

content

- ✓ mathematical concepts
  - topological spaces
  - continuous and smooth functions
  - homeo- and diffeomorphisms
  - manifolds
  - metric spaces
  - curvature
  - Riemannian surfaces
  - geodesic distances
  - Laplace-Beltrami operator
  - Gromov-Hausdorff distance
- ✓ concepts in action
  - surface correspondence
  - symmetry detection
  - intrinsic shape description

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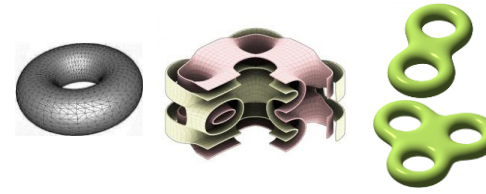
why topological spaces?

- ✓ to represent the set of observations made by the observer (e.g., neighbor, boundary, interior, projection, contour);
- ✓ to reason about stability and robustness



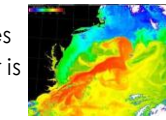
topological spaces

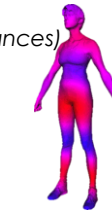
- ✓ a *topological space* is a set X together with a collection T of subsets of X, called *open sets*, satisfying the following axioms:
  1. $X, \emptyset \in T$
  2. any union of open sets is open
  3. any finite intersection of open sets is open

- ✓ the collection *T* is called a *topology* on *X*

why functions?

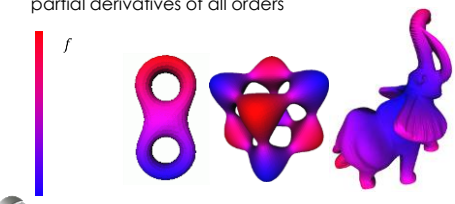
- ✓ to characterize shapes
- ✓ to measure shape properties
- ✓ to model what the observer is looking at
- ✓ to reason about stability
- ✓ to define relationships *(e.g., distances)*





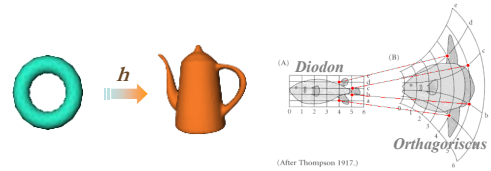
continuous and smooth functions

- let X, Y topological spaces, $f : X \to Y$ is *continuos* if for every open set $V \subseteq Y$ the inverse image $f^{-1}(V)$ is an open subset of X
- let X be an arbitrary subset of $\mathbb{R}^n$; $f : X \to \mathbb{R}^m$ is called *smooth* if $\forall x \in X$ there is an open set $U \subseteq \mathbb{R}^n$ and a function $F: U \to \mathbb{R}^m$ such that $F = f_{|X}$ on $X \cap U$ and F has continuous partial derivatives of all orders



homeo- & diffeo- morphisms

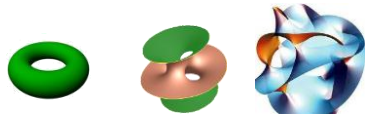
- a *homeomorphism* between two topological spaces X and Y is a continuous bijection $h: X \to Y$ with continuous inverse h^{-1}



- given $X \subseteq \mathbb{R}^n$ and $Y \subseteq \mathbb{R}^m$, if the smooth function $f: X \to Y$ is bijective and f^{-1} is also smooth, the function f is a *diffeomorphism*

why manifolds?

- to formalize shape properties
- to *ease* the analysis of the shape
  - measuring properties *walking on the shape*
  - look at the shape *locally* as if we were in our traditional euclidean space
  - to exploit *additional geometric* structures which can be associated to the shape



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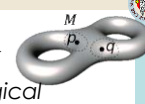
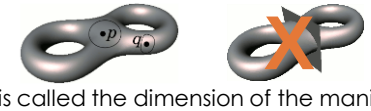
manifold

- manifold without boundary
  a *topological Hausdorff space* M is called a *k-dimensional topological manifold* if each point $q \in M$ admits a neighborhood $U_i \subseteq M$ homeomorphic to the open disk $D^k = \{x \in \mathbb{R}^k \, | |x| < 1\}$ and $M = \bigcup_{i \in \mathbb{N}} U_i$





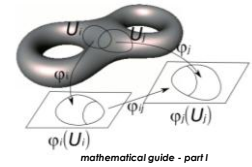
- k is called the dimension of the manifold

manifold

- manifold with boundary
  a topological Hausdorff space M is called a *k-dimensional topological manifold with boundary* if each point $q \in M$ admits a neighborhood $U_i \subseteq M$ homeomorphic either to the open disk $D^k = \{x \in \mathbb{R}^k \, | |x| < 1\}$ or the open half-space $\mathbb{R}^{k-1} \times \{y \in \mathbb{R} \mid y \geq 0\}$ and $M = \bigcup_{i \in \mathbb{N}} U_i$

smoothness and orientability

- transition functions
  let $\{(U_i, \varphi_i)\}$ an union of charts on a *k*-dimensional manifold M, with $\varphi_i: U_i \to D^k$. the homeomorphisms $\varphi_{i,j}: \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$ such that $\varphi_{i,j} = \varphi_j \cap \varphi_i^{-1}$ are called *transition functions*



smoothness and orientability

- smooth manifold

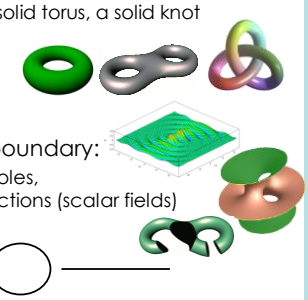
  a k-dimensional topological manifold with (resp. without) boundary is called a *smooth manifold* with (resp. without) boundary, if all transition functions $\varphi_{i,j}$ are smooth

- orientability

  a manifold M is called orientable is there exists an atlas $\{(U_i, \varphi_i)\}$ on it such that the Jacobian of all transition functions is positive for all intersecting pairs of regions

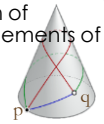
examples

- 3-manifolds with boundary:
  - a solid sphere, a solid torus, a solid knot
- 2-manifolds:
  - a sphere, a torus
- 2-manifold with boundary:
  - a sphere with 3 holes, single-valued functions (scalar fields)
- 1 manifold:
  - a circle, a line



metric space

- a *metric space* is a set where a notion of distance (called a metric) between elements of the set is defined



- formally,
  - a **metric space** is an ordered pair (X, d) where X is a set and d is a metric on X (also called distance function), i.e., a function

    $$d: X \times X \to \mathbb{R}$$

    such that $\forall x, y, z \in X$:
    - $d(x, y) \geq 0$; *(non-negative)*
    - $d(x, y) = 0$ iff $x = y$; *(identity)*
    - $d(x, y) = d(y, x)$; *(symmetry)*
    - $d(x, z) \leq d(x, y) + d(y, z)$ *(triangle inequality)*

what properties and invariants?

- how far are $\boldsymbol{p}, \boldsymbol{q}$ on X and $\boldsymbol{p'}, \boldsymbol{q'}$ on Y?

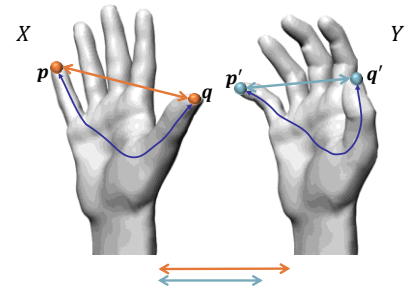
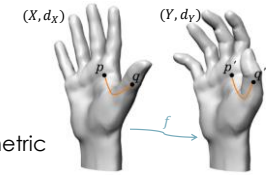


image partially from: Bronstein A. et al. PNAS 2006;103:1168-1172

isometries

- an *isometry* is a bijective map between metric spaces that preserves distances:

  $$f: X \to Y, d_Y(f(x_1), f(x_2)) = d_X(x_1, x_2)$$



- looking for the right metric space…
  - the Euclidean distance $d(x,y) = \sum_{i=1}^{n} \sqrt{(x_i - y_i)^2}$
  - geodesic distances, diffusion distances, …

image partially from: Bronstein A. et al. PNAS 2006;103:1168-1172

invariance and isometries

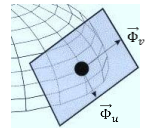
- a property invariant under isometries is called an *intrinsic* property

- examples:
  - the Gaussian curvature K
  - the first fundamental form
  - the geodesic distance
  - the Laplacian operator

parametric representation of surfaces

- regular parameterisation of a surface S:
  $\Phi: U \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3$
  - $\Phi(u,v) \in S, for\ all\ (u,v) \in U$
  - $\Phi(u,v) = \big(x(u,v), y(u,v), z(u,v)\big)$

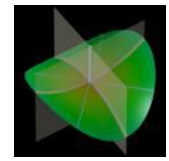
  such that

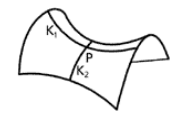


$$\begin{pmatrix} \frac{\partial x}{\partial u} \\ \frac{\partial y}{\partial u} \\ \frac{\partial z}{\partial u} \end{pmatrix} \times \begin{pmatrix} \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial v} \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

principal curvatures

- the principal curvatures measure the maximum and minimum bending of a surface at each point along lines defined by the intersection of the surface with planes containing the normal



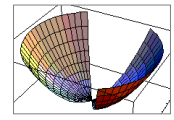


Gaussian and mean curvature

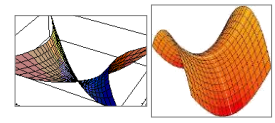
- given k_1 and k_2 the principal curvatures at a point surface
  - *Gaussian curvature* $K = k_1 k_2$
  - *mean curvature* $H = (k_1 + k_2)/2$
- according to the behavior of the sign of K, the points of a surface may be classified as
  - elliptic
  - hyperbolic
  - parabolic or planar

examples

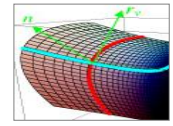
-



K>0



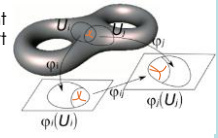
K<0



$K = 0, H \neq 0$

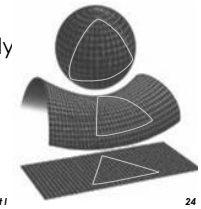
conformal structure

- a conformal structure is a structure assigned to a topological manifold such that angles can be defined
  - in the parameter plane the definition of angles is easy
  - to cover a manifold it could be necessary to consider many local coordinate systems with overlapping
  - if the transition function from one local coordinates to another is angle preserving, the angle value is independent of the choice of the local chart
  - if an atlas is such that all transition maps are conformal, angles can be consistently defined



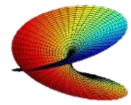
conformal structure and Riemann surfaces

- a conformal structure is the maximal conformal structure
- a topological surface with a conformal structure is called a *Riemann surface*
- *(uniformization)* any simply connected Riemann surface is either conformally equivalent to:
  - the open unit disk
  - the complex plane
  - the Riemann sphere



Riemannian surfaces

- a 2-manifold (real) can be turned into a Riemannian surface iff it is orientable and metrizable
- a Möbius strip, Klein bottle, projective plan do not admit a conformal structure



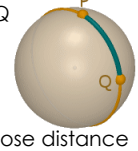
fundamental forms

- the *first fundamental form* is the inner product of the tangent space $S \subset E^3$ canonically induced from the dot product of $\mathbb{R}^3$
  - explicitly, if s is the Riemannian metric of S and Φ a regular surface parameterization
    $I = ds^2 = Edu^2 + 2Fdudv + Gdv^2$
    with $E = \left|\frac{\partial \Phi}{\partial u}\right|^2$, $F = \frac{\partial \Phi}{\partial u} \cdot \frac{\partial \Phi}{\partial v}$, $G = \left|\frac{\partial \Phi}{\partial v}\right|^2$
- the *second fundamental form* is the symmetric bilinear form on the tangent space
  $II = Ldu^2 + 2Mdudv + Ndv^2$
  with $L = \mathbf{N} \cdot \Phi_{u,u}$, $M = \mathbf{N} \cdot \Phi_{u,v}$, $N = \mathbf{N} \cdot \Phi_{v,v}$, and $\mathbf{N}$ normal vector
- in terms of fundamental forms the Gaussian curvature is

$$K = \frac{\det(II)}{\det(I)} = \frac{LN - M^2}{EG - F^2}$$

geodesic distance

- the arc length of a curve γ is given by $\int_\gamma ds$
- *minimal geodesics*: shortest path between two points on the surface
- *geodesic distance* between P and Q: length of the shortest path between *P* and Q
- geodesic distances satisfy all the requirements for a metric
- a Riemannian surface carries the structure of a metric space whose distance function is the geodesic distance



Laplace-Beltrami operator

- continuous case
  $\Delta f := \mathrm{div}(\mathrm{grad} f)$
- Laplacian eigenvalue problem
  $\Delta f = -\lambda f$
- orthonormal eigensystem
  $\mathcal{B} := \{(\lambda_i, \psi_i)\}_i \qquad \Delta\psi_i = \lambda_i \psi_i$
  $\lambda_0 \leq \lambda_1 \leq \ldots, \lambda_i \leq \lambda_{i+1} \ldots \leq +\infty$

discrete Laplacian operator

$$\Delta f(\mathbf{p}_i) := \frac{1}{d_i} \sum_{j \in N(i)} w_{ij} \left[f(\mathbf{p}_i) - f(\mathbf{p}_j) \right]$$

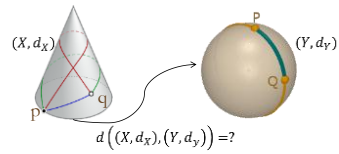
- $N(i)$ index set of 1-ring of vertex $\mathbf{p}_i$
- $f(\mathbf{p}_i)$ function value at vertex $\mathbf{p}_i$
- d_i mass associated with vertex $\mathbf{p}_i$
- w_{ij} edge weights

discrete geometric Laplacian

- Desbrun et al. (1999)
  $w_{ij} := \frac{\cot(\alpha_{ij}) + \cot(\beta_{ij})}{2} \qquad d_i := a(i)/3$
  - the cotangent weights take into account the angles opposite to edges, the masses take into account the area around vertices
- Meyer et al. (2002)
  $d_i := a_V(i)$
  - cotangent weights, masses considering the Voronoi area
- Belkin et al. (2003, 2008)
  - weights constructed using heat kernels
- Reuter et at. (2005, 2006)
  - weak formulation of the eigenvalue problem
    $\langle \Delta f, \varphi_i \rangle_{\mathcal{L}^2(\mathcal{M})} = -\lambda \langle f, \varphi_i \rangle_{\mathcal{L}^2(\mathcal{M})}$
    with φ_i cubic form functions

metrics between spaces

- the Gromov-Hausdorff distance poses the comparison of two spaces as the direct comparison of pairwise distances on the spaces
- equivalently, it measures the distortion of embedding one metric space into another



Gromov-Hausdorff distance

- let $(X, d_X), (Y, d_Y)$ be two metric spaces and $C \subset X \times Y$ a correspondence, the *distortion* of C is:

$$dis(C) = \sup_{(x,y),(x',y') \in C} |d_X(x, x') - d_Y(y, y')|$$

- the Gromov-Hausdorff distance is

$$d_{GH}(X, Y) = \frac{1}{2} \inf_C dis(C)$$

- variations: *Lp Gromov-Hausdorff* distances and *Gromov-Wasserstein* distances

properties

- the Gromov-Hausdorff distance is parametric with respect to the choice of metrics on the spaces X and Y
- common choices
  - Euclidean distance (estrinsic geometry)
  - geodesic distance (intrinsic geometry) or, alternatively, diffusion distance

$$d^2_{X,t}(x, y) = \sum_{i=0}^{\infty} e^{-2\lambda_i t} (\psi_i(x) - \psi_i(y))^2$$

where (λ_i, ψ_i) is the eigensystem of the Laplacian operator and t is time

references

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content

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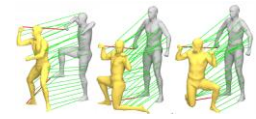
surface correspondence

- problem: finding correspondences between a discrete set of points on two surface meshes
  - *extrinsic* vs *intrinsic* correspondence

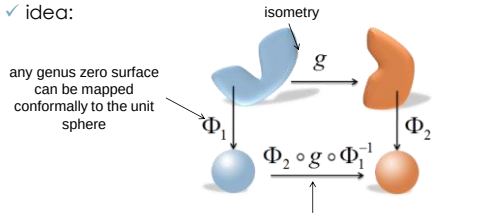


- why: **attribute transfer, surface tracking, shape analysis (brain imaging)**, …

Courtesy of Michael Bronstein

intrinsic correspondence [Lipman and Funkhouser 2009]

- looking for an **intrinsic** correspondence means finding corresponding points such that the mapping between them is close to an **isometry**
- idea:

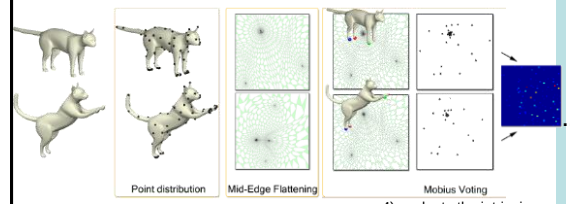


1-1 and onto conformal map of a sphere to itself (Mobius map): uniquely defined by three corresponding points

intrinsic correspondence [Lipman and Funkhouser 2009]

1) sampling points: local maxima of Gauss curvature & (geodesically) farthest point algorithm

3) compute the Möbius transformation that aligns a triplet in the common domain

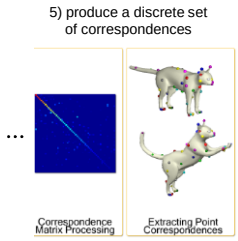


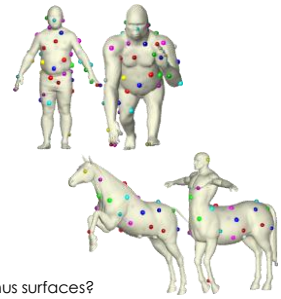
2) discrete conformal flattening to the extended complex plane

4) evaluate the intrinsic deformation error and build a fuzzy correspondence matrix

intrinsic correspondence [Lipman and Funkhouser 2009]

5) produce a discrete set of correspondences

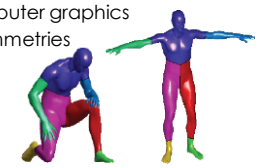




- pay attention to…
  - what about higher genus surfaces?
  - drawbacks of the discrete (linear) flattening technique

symmetry detection

- problem: detecting symmetries on a surface mesh
  - well studied problem in perceptual psychology, computer vision, computer graphics
  - *extrinsic* vs *intrinsic* symmetries



- why: compression, completion, matching, beautification, alignment…

Courtesy of Michael Bronstein

global intrinsic symmetry detection [Kim et al. 2010]

- looking for **intrinsic** symmetry transformations means finding isometric transformations that map a surface onto itself (**self-isometries**);

  (cf. the correspondence finding problem)
  - input: *M* orientable, genus zero surface
  - objective: $f: M \to M$ intrinsic symmetry
- orientation-preserving isometries are related to conformal maps; orientation-reversing isometries are related to anti-conformal maps
- ideas similar to the work just seen: here the anti-Möbius group (Möbius maps plus Möbius maps composed with a reflection) comes into play

global intrinsic symmetry detection [Kim et al. 2010]

1) sampling points: symmetry invariant sets S1 (coarse) and S2 (dense)

- IDEA: given a symmetry invariant function $\Phi: M \to R$, the set *S* of its critical points is a symmetry invariant set that satisfies $f(S) = S$
- IDEA: the Average Geodesic Distance (AGD) and the Minimal Geodesic Distance (MGD) are symmetry invariant functions

$$\Phi_{agd}(p) = \int_{\mathcal{M}} d_g(p,q) d\mathrm{vol}_{\mathcal{M}}(q)$$

$$\mathcal{S}_1 = \{p \in \mathcal{M} \mid \nabla|_p \Phi_{agd} = 0\}$$

$$\Phi_{mgd}(p) = \Phi_{mgd}(p; \mathcal{S}_1) = \min_{q \in \mathcal{S}_1} d_g(p,q)$$

$$\mathcal{S}_2 = \{p \in \mathcal{M} \mid \nabla|_p \Phi_{mgd} = 0\}$$

global intrinsic symmetry detection [Kim et al. 2010]

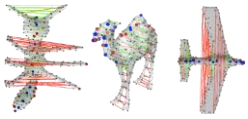
1) sampling points: symmetry invariant sets S1 (coarse) and S2 (dense)

2) discrete conformal flattening to the extended complex plane

3) compute anti-Möbius transformations that align triplets and quadruplets of S1 in the common domain

4) apply th transformations to S2 and evaluate the intrinsic deformation error (based on geodesics)

5) use the best transformation to extract correspondences within the symmetry invariant set S2

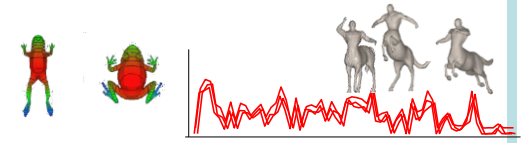


pay attention to…

✓symmetry invariant functions have to be smooth!
✓what about partial and small symmetries?

deformable shape characterization

✓ problem: describing intrinsic shape properties, and deriving signatures



Courtesy of Michael Bronstein

✓ why: shape registration, global and partial matching, …

intrinsic shape description [Sun et al. 2009]

✓ *heat equation*, governing the distribution of heat from a source on a surface X; initial conditions: heat distribution at time $t = 0$:

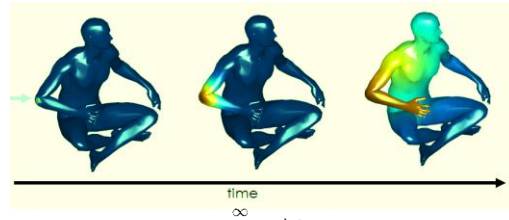
$$\left(\Delta_X - \frac{\partial}{\partial t}\right) u = 0$$

✓ the heat kernel is a fundamental solution of the heat equation with point heat source at x (and heat value at y after time t)

$$K_t(x,y) = \sum_{i=0}^{\infty} e^{-\lambda_i t}\phi_i(x)\phi_i(y)$$

$$\approx \sum_{i=0}^{N} e^{-\lambda_i t}\phi_i(x)\phi_i(y)$$

intrinsic shape description [Sun et al. 2009]



$$K_t(x,y) = \sum_{i=0}^{\infty} e^{-\lambda_i t}\phi_i(x)\phi_i(y)$$

$$\approx \sum_{i=0}^{N} e^{-\lambda_i t}\phi_i(x)\phi_i(y)$$

Courtesy of Michael Bronstein

intrinsic shape description [Sun et al. 2009]

✓ heat kernel signature as a point description over the temporal domain:

$$HKS(x): \mathbb{R}^+ \to \mathbb{R}, \qquad HKS(x,t) = k_t(x,x)$$

✓ multiscale, informative, intrinsic, localized sensitivity to topological noise

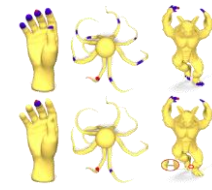
✓ distance between signatures at scale $[t_1, t_2]$:

$$d_{[t_1,t_2]}(x,x') = \left(\int_{t_1}^{t_2} \left(\frac{|k_t(x,x) - k_t(x',x')|}{\int_M k_t(x,x)dx}\right)^2 d\log t\right)^{1/2}$$

Courtesy of Michael Bronstein

intrinsic shape description [Sun et al. 2009]

feature point detection



localizing repeated structures at differet scales

✓ pay attention to…
- multiscale… but how to handle time scale?
- discretization and robustness issues

references

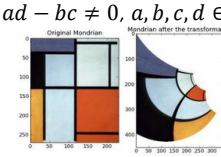
- Kim, V., Lipman, Y., Chen, X., and Funkhouser, T. 2010. Mobius transformations for global intrinsic symmetry analysis. *Computer Graphics Forum 29(5)(Symposium on Geometry Processing)*, 1689-1700
- Lipman, Y., and Funkhouser, T. 2009. Mobius voting for surface correspondence. *ACM Transactions on Graphics* 28(3) *(SIGGRAPH)*
- Sun, J., Ovsjanikov, M., and Guibas, L. 2009. A concise and provably informative multi-scale signature based on heat diffusion. In *Proc. of the Symp. on Geometry Processing*, Eurographics Association, Aire-la-Ville, Switzerland, Switzerland, SGP '09,1383–1392
- Van Kaick O., Zhang H., Harmaneh G., Cohen-Or D.. A survey on shape correspondence, Computer Graphics Forum, 30(6):1681-1707, 2011
- T. Liu, V. G. Kim, T. Funkhouser.. Finding Surface Correspondences Using Symmetry Axis Curves Computer Graphics Forum, 31(5):1607-1616, 2012

Möbius transformation

- is a rational function $f: \mathbb{C} \cup \{\infty\} \rightarrow \mathbb{C} \cup \{\infty\}$:

$$f(z) = \frac{az+b}{cz+d}$$

such that $ad - bc \neq 0$, $a, b, c, d \in \mathbb{C}$



  - is a bijective conformal map of the Riemann sphere to itself
  - also known as *homographic*, *linear fractional*, *bilinear*, or *fractional linear transformation*