Mathematical Tools for 3D Shape Analysis and Description

SGP 2013 *Graduate School*

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outline

- ✓ motivation
- ✓ mathematics and shape analysis challenges *(11:35–11:45)*
  - shape properties and invariants
  - similarity between shapes
- ✓ tools and concepts, part I *(11:45-12:15)*
  - topological spaces, functions, manifolds
  - metric spaces, isometries, curvature, geodesics
  - Gromov-Hausdorff distance
  - concepts in action
- ✓ tools and concepts, part II *(14:00-15:00)*
  - basics on topology, homology and Morse theory
  - natural pseudo-distance
  - concepts in action
- ✓ conclusions *(15:00-15:15)*

where are we now?

- ✓ technology today
  - plenty of 3D acquisition techniques
  - hardware for visualizing 3D on the desktop
  - computer networks: fast connections, low cost
  - 3D printers: not only mock-ups but even end products

rendering, acquiring, transmitting, "materializing" 3D content is now feasible in specialized as well as unspecialized contexts

3D media

- ✓ professionals
  - Product Modeling & Design
  - Cultural Heritage
  - Gaming
  - Spatial Data
  - Simulation
  - Medicine
  - Bioinformatics
  - Architecture
  - Archaeology
- ✓ non professionals
  - 3D social networking
  - fabbing
  - ...

… how to analyse, describe, process, organize, navigate, filter, share, re-use and re-purpose, this large amount of complex content ?

reasoning about shape, similarity, semantics

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mathematics and shape analysis challenges

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shape and geometry

- "… all the geometrical information that remains when location, scale, and rotational effects are filtered out from an object" [Kendall 1977]

shape and *similarity*

- "…the form of something by which it can be seen (or felt) different by something else" [Longman Dictionary of Contemporary English]

that sounds nice but… what do "similar" and "different" mean?

shape, similarity & the observer

- things possess a shape for the observer, in whose mind the association between the perception and the existing conceptual models takes place [Koenderink 1990]

understanding, reasoning, similarity is a cognitive process, depending on the observer and the context

shape, similarity & the observer

- **things** possess a **shape** for the **observer**, in whose mind the association between the **perception** and the existing **conceptual** models takes place [Koenderink 1990]

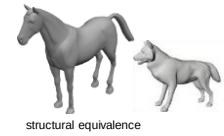
understanding, reasoning, similarity is a cognitive process, depending on the observer and the context

shape and view points

Guido Moretti's sculptures

objects and similarity

geometric congruence



structural equivalence

functional equivalence

semantic equivalence

objects and similarities

geometric congruence

structural equivalence

functional equivalence

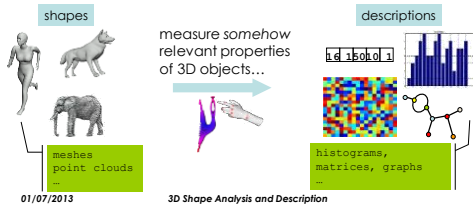
semantic equivalence

mathematics: shape description and similarity

- similar shapes with **respect to what**?
  - shape descriptions, to code the aspects of shapes to be taken into account and manage the complexity of the problem
- **similarity** in what sense ?
  - transformations among the shapes that we consider irrelevant to the assessment of the similarity
    - invariants or properties

shape and description

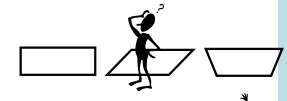
- shape descriptions reduce the complexity of the representation; their choice depends on
  - **type** of shapes and their variability/complexity
  - **invariants** or properties



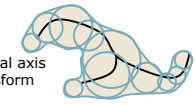
shape descriptions

- different shapes should have different descriptions
  - different enough to discriminate among shapes
- a shape may not be entirely reconstructed from its description

example # edges = 4



edge length and angle



medial axis transform

what's invariance?

- invariance = the descriptor does not change for a given object under a class of transformations
- a property P is invariant to a transformation T applied to an object O iff

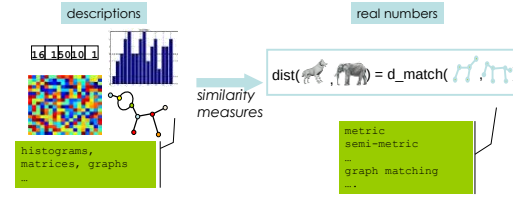
$$P(T(O)) = P(O)$$

example

boundary length

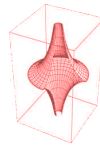
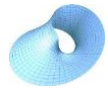
shape descriptions and similarity

- *similarity* in what sense ?
  - defining appropriate *similarity measures* between shape descriptions



things are not that easy…

- to deal with the *complexity* at a hand…
- we need tools to reason about
  - connectivity, interior, exterior and boundary
  - measuring shape properties and invariants
  - well-posedness
  - robustness and stability
  - distance and proximity
  - etc…

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tools and concepts, part I

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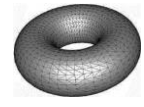
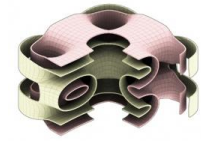
content

- tools and concepts
  - topological spaces
  - continuous and smooth functions
  - homeo- and diffeomorphisms
  - manifolds
  - transformations
  - metric spaces
  - intrinsic properties
    - curvature
    - conformal structure
    - geodesic distances
    - Laplace-Beltrami operator
  - Gromov-Hausdorff distance
- concepts in action

why topological spaces?

- to represent the set of observations made by the observer (e.g., neighbor, boundary, interior, projection, contour);
- to reason about stability and robustness

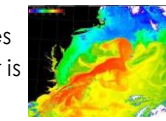
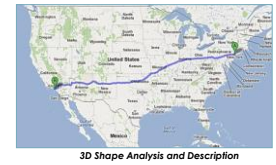
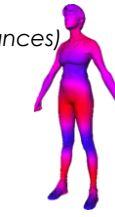
topological spaces

- a *topological space* is a set X together with a collection T of subsets of X, called *open sets,* satisfying the following axioms:
  1. $X, \emptyset \in T$
  2. any union of open sets is open
  3. any finite intersection of open sets is open

- the collection *T* is called a *topology* on *X*

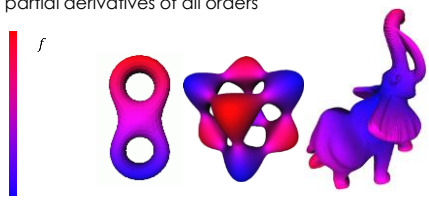
why functions?

- to characterize shapes
- to measure shape properties
- to model what the observer is looking at
- to reason about stability
- to define relationships *(e.g., distances)*

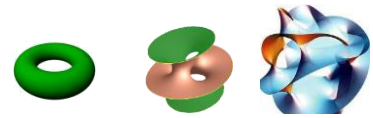
continuous and smooth functions

- let X, Y topological spaces, $f : X \to Y$ is *continuos* if for every open set $V \subseteq Y$ the inverse image $f^{-1}(V)$ is an open subset of X
- let X be an arbitrary subset of $\mathbb{R}^n$; $f : X \to \mathbb{R}^m$ is called *smooth* if $\forall x \in X$ there is an open set $U \subseteq \mathbb{R}^n$ and a function $F: U \to \mathbb{R}^m$ such that $F = f_{|X}$ on $X \cap U$ and F has continuous partial derivatives of all orders



why manifolds?

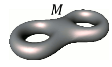
- to formalize shape properties
- to *ease* the analysis of the shape
  - measuring properties *walking on the shape*
  - look at the shape *locally* as if we were in our traditional euclidean space
  - to exploit *additional geometric* structures which can be associated to the shape



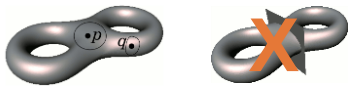
Images courtesy of D. Gu and Jbourjai on Wikimedia Commons

manifold

- manifold without boundary



a *topological Hausdorff space* M is called a *k-dimensional topological manifold* if each point $q \in M$ admits a neighborhood $U_i \subseteq M$ homeomorphic to the open disk $D^k = \{x \in \mathbb{R}^k \,|\, |x| < 1\}$ and $M = \bigcup_{i \in \mathbb{N}} U_i$



- k is called the dimension of the manifold

manifold

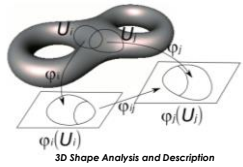
- manifold with boundary

a topological Hausdorff space M is called a *k-dimensional topological manifold with boundary* if each point $q \in M$ admits a neighborhood $U_i \subseteq M$ homeomorphic either to the open disk $D^k = \{x \in \mathbb{R}^k \,|\, |x| < 1\}$ or the open half-space $\mathbb{R}^{k-1} \times \{y \in \mathbb{R} \mid y \geq 0\}$ and $M = \bigcup_{i \in \mathbb{N}} U_i$

smoothness and orientability

- transition functions

let $\{(U_i, \varphi_i)\}$ an union of charts on a *k*-dimensional manifold M, with $\varphi_i : U_i \to D^k$. the homeomorphisms $\varphi_{i,j} : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$ such that $\varphi_{i,j} = \varphi_j \cap {\varphi_i}^{-1}$ are called *transition functions*



smoothness and orientability

- smooth manifold

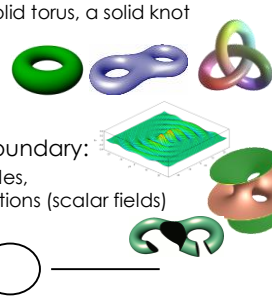
a k-dimensional topological manifold with (resp. without) boundary is called a *smooth manifold* with (resp. without) boundary, if all transition functions $\varphi_{i,j}$ are smooth

- orientability

a manifold M is called orientable is there exists an atlas $\{(U_i, \varphi_i)\}$ on it such that the Jacobian of all transition functions is positive for all intersecting pairs of regions

examples

- 3-manifolds with boundary:
  - a solid sphere, a solid torus, a solid knot
- 2-manifolds:
  - a sphere, a torus
- 2-manifold with boundary:
  - a sphere with 3 holes, single-valued functions (scalar fields)
- 1 manifold:
  - a circle, a line



metric space

- a *metric space* is a set where a notion of distance (called a metric) between elements of the set is defined
- formally,
  - a **metric space** is an ordered pair (X, d) where X is a set and d is a metric on X (also called distance function), i.e., a function

    $$d: X \times X \rightarrow \mathbb{R}$$

    such that $\forall x, y, z \in X$:
    - $d(x, y) \geq 0$; *(non-negative)*
    - $d(x, y) = 0$ iff $x = y$; *(identity)*
    - $d(x, y) = d(y, x)$; *(symmetry)*
    - $d(x, z) \leq d(x, y) + d(y, z)$ *(triangle inequality)*

what properties and invariants?

- is it possible to transform the space X into Y?
- how to formalize that?

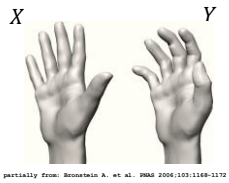


image partially from: Bronstein A. et al. PNAS 2006;103:1168-1172

tranformations

- congruence
  - two objects are congruent if one can be transformed into the other by rigid movements (translation, rotation, reflection – **not scaling**)

transformations

- similarity
  - two geometrical objects are called *similar* if one can be obtained by the other by uniform stretching . Formally, a *similarity* of a Euclidean space S is a function $f: S \rightarrow S$ that multiplies all distances by the same positive scalar r, so that:

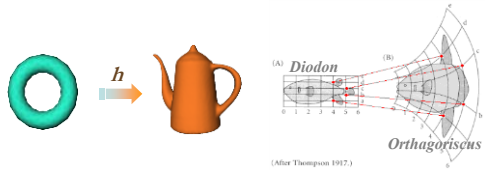
    $$d(f(x), f(y)) = rd(x, y), \forall x, y \in S$$

transformations

- affinity
  - it preserves collinearity, i.e. maps parallel lines into parallel lines and preserve ratios of distances along parallel lines
  - it is equivalent to a linear transformation followed by a translation

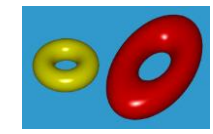
homeo- & diffeo- morphisms

- a *homeomorphism* between two topological spaces X and Y is a continuous bijection $h: X \to Y$ with continuous inverse h^{-1}



- given $X \subseteq \mathbb{R}^n$ and $Y \subseteq \mathbb{R}^m$, if the smooth function $f: X \to Y$ is bijective and f^{-1} is also smooth, the function f is a *diffeomorphism*

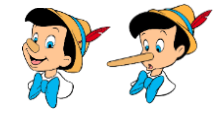
transformations and similarities



affine transformation

image from http://cse.taylor.edu/~btoll/s99/424/res/mtu/Notes/geometry/geo-tran.htm

isometric transformation



"locally-affine" transformation

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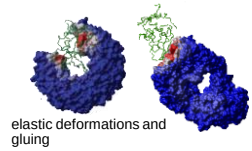


elastic deformations and gluing

transformations and metric spaces

- how far are $\boldsymbol{p}, \boldsymbol{q}$ on X and $\boldsymbol{p}', \boldsymbol{q}'$ on Y?

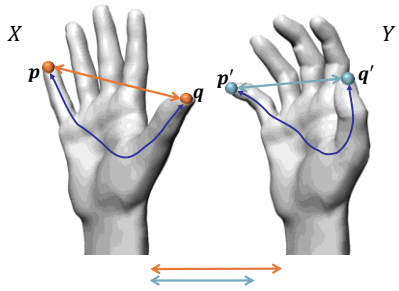
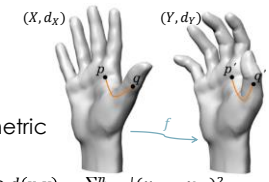


image partially from: Bronstein A. et al. PNAS 2006;103:1168-1172

isometries

- an *isometry* is a bijective map between metric spaces that preserves distances:

$$f: X \to Y, d_Y\big(f(x_1), f(x_2)\big) = d_X(x_1, x_2)$$



- looking for the right metric space…
  - the Euclidean distance $d(x,y) = \sum_{i=1}^{n} \sqrt{(x_i - y_i)^2}$
  - geodesic distances, diffusion distances, …

image partially from: Bronstein A. et al. PNAS 2006;103:1168-1172

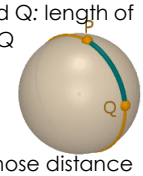
invariance and isometries

- a property invariant under isometries is called an *intrinsic* property

- examples:
  - the Gaussian curvature K
  - the first fundamental form
  - the geodesic distance
  - the Laplace-Beltrami operator

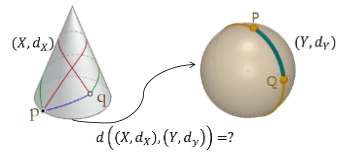
geodesic distance

- the arc length of a curve γ is given by $\int_\gamma ds$
- *minimal geodesics*: shortest path between two points on the surface
- *geodesic distance* between P and Q: length of the shortest path between *P* and *Q*
- geodesic distances satisfy all the requirements for a metric
- a Riemannian surface carries the structure of a metric space whose distance function is the geodesic distance



metrics between spaces

- the Gromov-Hausdorff distance poses the comparison of two spaces as the direct comparison of pairwise distances on the spaces
- equivalently, it measures the distortion of embedding one metric space into another



Gromov-Hausdorff distance

- let $(X, d_X), (Y, d_Y)$ be two metric spaces and $C \subset X \times Y$ a correspondence, the *distortion* of C is:

$$dis(C) = \sup_{(x,y),(x',y') \in C} |d_X(x, x') - d_Y(y, y')|$$

- the Gromov-Hausdorff distance is

$$d_{GH}(X, Y) = \frac{1}{2} \inf_C dis(C)$$

- variations: *Lp Gromov-Hausdorff* distances and *Gromov-Wasserstein* distances

properties

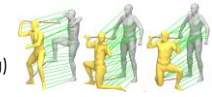
- the Gromov-Hausdorff distance is parametric with respect to the choice of metrics on the spaces X and Y
- common choices
  - Euclidean distance (estrinsic geometry)
  - geodesic distance (intrinsic geometry) or, alternatively, diffusion distance

$$d^2{}_{X,t}(x, y) = \sum_{i=0}^{\infty} e^{-2\lambda_i t} (\psi_i(x) - \psi_i(y))^2$$

where (λ_i, ψ_i) is the eigensystem of the Laplacian operator and t is time

concepts in action

- surface correspondence
  - attribute transfer,
  - surface tracking,
  - shape analysis (brain imaging)
  - …
- symmetry detection
  - compression,
  - completion,
  - matching,
  - beautification,
  - alignment
  - …



…stay tuned….
see the Michael Bronstein's talk

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any question?

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